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Optimal Control System to Solve Dynamic Supply Chain Problem

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Abstract

The work considers a control problem of production schedule to get optimal schedule to minimize the cost of supply chain (SC). The model is linear time invariant system. The model is solved using open loop optimal control (OLOC) and closed loop optimal control (CLOC).
Keywords: Optimal control system; Weibull distribution; Open loop optimal control; Closed loop optimal control.

1. Introduction

Optimal control System is a very useful mathematical field with many applications in both science and engineering. Recent years optimal Control system becomes famous tool to solve a problems of dynamic nature in management science and operations research [1] applied optimal controls to maximize customers satisification and reduce operating cost. Sethi and Thompson [2] modeled dynamical systems by sets of differential equations. Benhadid and Lotfi [3] described how to solve production inventory problem with deterioration item. Chaudhary, *et al.* [4] introduced optimal control approach to find solution of demand content Taher and Naglaa [5] described how to control SC problem with Weibull distribution. Zaher and Zaki [6] studied a model to manage three stages SC. In this paper research, We describe both approaches which we studied in pervious papers ,we solved supply chain problem using both open loop optimal control system using principle of Pontryagin and feed back control system solving the matrix differential equation of Riccati (DRE) [7], [2], [8].

2. Assumptions and Notations

The following assumptions are considered to build the model.

T: Final time (length of time horizon).

h_1 : Holding cost. of Company

h_2 : Holding cost of Supplier

c : The cost of production

$\hat{p}$: Required production belongs to vendor

$\hat{x}_1$: Desired level of buyer.

$\hat{x}_2$: wanted scale of retailer.

D (t): rate of quantity demanded .

$\theta_1(t)$: Rate of spoiling in company store.

$\theta_2(t)$: Rate of spoiling in suppliers stores.

Outputs

$p(t)$: Manufacture actual rate.

$x_1(t)$: Level of stock related to company.

$x_2(t)$: Level of stock related to buyer.

The performance index of the model can be written as:

$$\text{Minimize } Z = \int_0^T \frac{1}{2} \ldots \frac{1}{2}\left(x_2(t)-\hat{x}_2\right)^2 + \frac{c}{2} \ldots$$

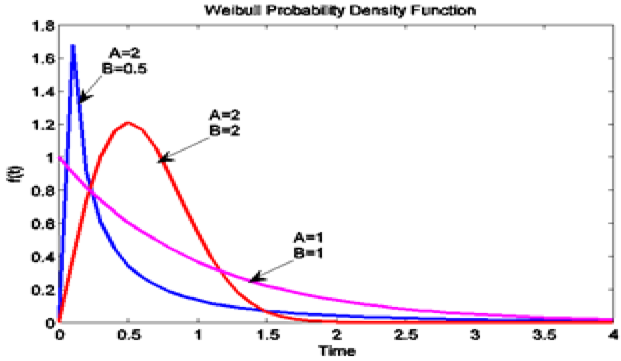
Such that

$$\dot{x}_1 = p(t) - d_1(t) - \theta_1(t)x_1, \quad x_1(0) = x_{10}$$

$$\dot{x}_2 = -d_2(t) - \theta_2(t)x_2, \quad x_2(0) = x_{20}$$

The average deterioration related to the time is in the form: [1]

Fig-1. Probability Density Function



$$\theta(t) = \frac{f(t)}{1-F(t)} = AB\,t^{B-1} \qquad t \quad 0 \tag{5}$$

Fig.1 shows the probability density function of Weibull distribution at different values of the parameters A, B.

3. The Mathematical Model and Analysis

3.1 Open Loop Optimal Control

Starting is by applying Pontryagin maximum principle [7], [9], [8], [2]. The Hamiltonian is written as follows:

$$H = \frac{1}{2} h_1 \left(x_1(t) - \hat{x}_1\right)^2 + \frac{1}{2} h_2 \left(x_2(t) - \hat{x}_2\right)^2 + \frac{c}{2}\left(p(t) - \hat{p}\right)^2 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 \tag{6}$$

To find the optimal control, we maximize by differentiating H equation (6) with respect to p and setting the result to zero $\frac{\partial H}{\partial p} = 0$ (necessary condition for p to be optimal control)

$$p(t) = \hat{p} + \frac{\lambda_2}{c} \tag{7}$$

State and Costate Equations:

$\dot{x}_1 = \frac{\partial H}{\partial \lambda_1}$ State equation

$$\dot{x}_1 = p(t) - d(t) - \theta_1(t)x_1 \quad (8)$$

Substitute equation (7) into equation (8)

$$\dot{x}_1 = p^* + \frac{\lambda_2}{c} - d(t) - \theta_1(t)x_1 \quad (9)$$

$\dot{x}_2 = \frac{\partial H}{\partial \lambda 2}$ State equation

$$\dot{x}_2 = -d(t) - \theta_2(t)x_2 \quad (10)$$

$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1}$ Costate equation

$$\quad (11)$$

$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2}$ Costate equation

$$\dot{\lambda}_2 = h_2(x_2 - x_2^*) + \lambda_2\theta_2 \quad (12)$$

Solve the previous 4 differential equations (9),(10),(11),(12) with boundary condition equation (13)

$$\lambda_1(T) = 0, \; \lambda_2(T) = 0, \; x_1(0) = x_{10}, \; x_2(0) = x_{20} \quad (13)$$

to get $x_1(t)$, $x_2(t)$, $\lambda_1(t)$ and $\lambda_2(t)$

According to (Naidu 2003) this type of optimal control problem is known as fixed final time and free final state system.

3.2. Closed Loop Optimal Control

It is clear those equations (1), (2) and (3) can be changed by simple substitution to become identical to general form of linear quadratic regulator system defined in ([7], [8], [10] Minimize j =

$$\int_0^T \left(\frac{1}{2}\left(x'(t)Q(t)x(t) + u'(t)R(t)u(t)\right)\right) dt \quad (14)$$

Subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (15)$$

The boundary conditions are

$x(0) = x_0$, T is fixed and $x(T)$ Is free.

The solution of equations (14) and (15) is as flowing:

***Step 1**: solve the matrix differential Riccati equation [7, 8].*

$$\dot{M}(t) = -M(t)A(t) - A'(t)M(t) - Q(t) + M(t)B(t)R^{-1}(t)B'(t)M(t) \quad (16)$$

With Final Condition

M (*t*=*T*) = *F* (*T*)=0

The analytical Solution of Differential Riccati Equation (DRE) is given by Kuo [7] and Naidu [8].

$$H(\tau) = -e^{-\Lambda(T-t)} \left[w_{22} - w_{12}\right]^{-1} \left[w_{21} - F w_{11}\right] e^{-\Lambda(T-t)} \quad (17)$$

$$M(t) = \left[w_{21} + w_{22} H(\tau)\right] \left[w_{11} + w_{12} H(\tau)\right]^{-1} \quad (18)$$

where

$\tau = T - t$

Λ : Diagonal matrix contains the eigen values of matrix

$$\begin{bmatrix} A(t) & -B(t)R(t)B^{T}(t) \\ -Q(t) & -A'(t) \end{bmatrix}$$

W: Matrix of eigen vectors correspond to diagonal matrix Λ be defined as

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

Step 2: Solve optimal state x form

$$\dot{x}(t) = [A(t) - B(t)R^{-1}(t)B' M(t)]x(t) \quad (19)$$

With initial condition $x(0) = x_0$.

Step 3: obtain the optimal control $u(t)$ as

$$u(t) = (-R^{-1}(t)B'(t)M(t))x(t) \quad (20)$$

4. Numerical Example

The presumed parameters of this example assume as in table 1.

Table-1. Parameters Given

Parameters	c	$\hat{p}$	h_1	h_2	$\hat{x}_1$
Values	9	15	10	12	40
Parameters	$\hat{x}_2$	X10	X20	A	B
Values	25	10	15	1	1

Where

c, h_1, h_2 are the production cost, the company holding cost and the suppliers holding cost per unit product. Respectively

$\hat{p}$, $\hat{x}_1$.$\hat{x}_2$ are the production goal rate, inventory goal level of the company and inventory goal level of the suppliers.

4.1. Solution using Open Loop Optimal Control

First step is to solve set of differential equations (9),(10),(11),(12) with boundary condition (13) using Toolbox of MATLAB version (7).

Second step is to substitute in equation (7) to get optimal Production.

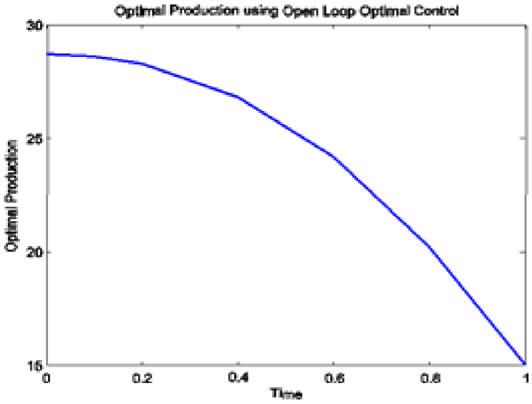
The solution for optimal production is plotted in Fig.2.

The values of inventory levels $x_1(T)$, $x_2(T)$, rate of production $p(T)$ and objective function are summarized in table 2. i.e these values are calculated at time $t = T = 1$ at end of time horizon.

Table-2. Results at t=T =1

Parameters	$x_1(T)$	$x_2(T)$	$p(T)$	j
Values	15	2	15	5716

Figure-2. Optimal Production using OLOC



4.2. Solution using Closed Loop Optimal control

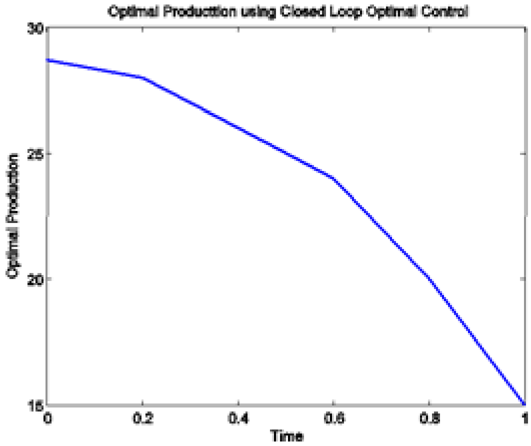
First step: various quantities is identified as:

$$A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix};\quad B=\begin{bmatrix}1\\ 0\end{bmatrix};\quad Q=\begin{bmatrix}10 & 0\\ 0 & 12\end{bmatrix};$$

$$F=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix};\quad R=9;\quad T=1$$

Second step: The solution of matrix differential Riccati is exist in equation (18), solve equation (19) for optimal states ($x1$ (t), $x2$ (t)) and optimal control ($u(t)$)in equation (20) using MATLAB. The optimal production as function of time is plotted in Fig 3.

Figure-3. Optimal Production using CLOC.



The values of inventory levels $x(T)$, $x(T)$, rate of production $p(T)$ and objective function j are nearly the same as table 2. The optimal production rate which is obtained from using close loop optimal control is identical to using open loop optimal control but the closed loop optimal control is more stable and simpler than open loop optimal control.

5. Conclusions

This Study has studied the solution of dynamic supply chain model which consists of company and suppliers and customers focusing in using engineering control system to find optimal manufacture schedule that decrease the cost of the model. The model can be extended to become optimal control system with infinite time horizon case and the frequency response of the system can be studied, we can make comparison between local minimum and steady state value of the system.

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