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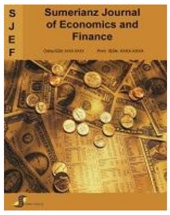

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The Sharpe Ratio With Skewness

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Abstract

We propose an investment choice model to improve existing risk-adjusted performance measures by incorporating portfolio skewness. Building on a partial differential equation analysis, we demonstrate that a risk-adjusted performance measure maximizes investor expected utility. To evaluate the Sharpe ratio with skewness adjustment for all investors, we price second-order returns using first-order returns and show that the resulting ratio is the sum of the Sharpe ratio due to first-order returns and the Sharpe ratio due to second-order returns. We further show that the Sharpe ratio with skewness adjustment increases with return mean, decreases with volatility, and non-decreases with skewness. Finally, we test the Sharpe ratio with skewness using 299 market indexes including country indexes such as the MSCI China a index and S and P 500. The test reveals that the Sharpe ratio with skewness can accurately assess performance of wide-range assets. Our model anchors a risk-adjusted performance measure to an investment choice.

Keywords: Sharpe ratio; Sharpe ratio with skewness; portfolio skewness; Risk-adjusted performance measure; Portfolio analytics; Portfolio construction.

1. Introduction

A risk-adjusted performance measure (RAPM) is a statistic to summarize asset (or portfolio) performance and risk. It is usually a ratio of a return to a risk such as the Sharpe ratio (Sharpe, 1966) [1]. This kind of RAPM is the portfolio performance when the portfolio exposes a unit of risk. A RAPM is used to evaluate and compare portfolio performance, construct a portfolio, and make an investment decision. The Sharpe ratio is one of most widely used RAPMs in the investment industry.

Investors have many RAPMs to select for their investments. These RAPMs have derived or proposed based on economic rationales or utility functions. They include the classical ones such as the Sharpe ratio, Treynor ratio Treynor (1965), and Jensen (1968), as well as the non-classical ones such as the conditional Sharpe ratio (Gregoriou and Gueyie, 2003), Omega (Shadwick and Keating, 2002), Lambda (Kaplan, 2005), a modified Sortino ratio (Pedersen and Satchell, 2002), and others (please see (Agarwal and Naik, 2004; Burke, 1994; Dowd, 2000; Kaplan and Knowles, 2004; Kestner, 1996; Sharma, 2004; Sortino and Price, 1994; Young, 1991).

There are cases that an investor picks an improper RAPM for an investment. For example, an investor invests in a relative return product, i.e., the performance and risk of the product are measured in terms of excess return and tracking error over a market index such as the S&P 500. However, she sometimes uses the Sharpe ratio as a RAPM to assess the investment. It is known that the Sharpe ratio is suitable for evaluating total return products which are measured by their total return and volatility. In other cases, an investor choose a right RAPM for an investment, but she receives inaccurate results. For example, an investor wants to invest in a total return product. Two market index products are available for her investment and each initial investment is required to invest $1 million. The two indexes will have the same Sharpe ratio of 0.83, but different normalized skewness of returns[2]: -0.17 vs. -0.10. The indexes are expected to separately accumulate $23.8 million and $26.9 million over a long period. As a result, the first product will make $3.1 less than the second product mainly due to its more negative return skewness than the second index[3]. The example clearly shows that the investor inaccurately value her potential investment without addressing return skewness even the Sharpe ratio is a right RAPM for total return products[4].

[1] The Sharpe ratio is the ratio of an excess return over a risk-free rate to volatility of a portfolio.

[2] The normalized skewness is defined by the expression: $\left[E\left(r_p - E\left(r_p\right)\right)^3\right] / \sigma\left(r_p\right)^3$, where $E\left(r_p\right)$ is the expected return and $\sigma\left(r_p\right)$ is the portfolio volatility.

[3] The data of two indexes come from Table 1, index 6 and index 7.

[4] An investor may inaccurately use the Sharpe ratio to evaluate any products linked to the S&P 500 as the market index has monthly return skewness of -0.419 from 1950 to 2016, and its skewness varies from different periods. Additionally, the Chinese market has better return normality than the US market. The Chinese index with 300 largest market-cap stocks, CS300, has return skewness of -0.013 from 2002 to 2016.

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To tackle an improper use of a RAPM, we link a RAPM to an investor's investment choice. An investment choice is a part of an investment decision process. It is related to decisions such as how investors fund their investments, what kinds of performance and risk they are willing to take, and which financial market (or product) they want to invest. An investment choice decides an investor's expected utility function. This paper extends the investment choice theory (Xiang *et al.*, 2012) to include more than one risk measure. In this theory, we derive a partial differential equation by maximizing the utility function associated with an investment choice. We show that a RAPM, which is suitable to the investment, must be the solution to the partial differential equation. The theory guarantees that an investor always choose a proper RAPM for her investment.

To address using a RAPM to inaccurately evaluate an investment, we propose the Sharpe ratio with skewness, that is the Sharpe ratio adjusted upward (downward) when returns have positive (negative) skew. We derive the Sharpe ratio with skewness from two perspectives. First, we show that a RAPM is the Sharpe ratio with skewness if an investor cares only return mean, volatility, and skewness of an investment. In this case, the Sharpe ratio with skewness is a solution to a partial differential equation linked to the investor's investment choice. It includes a parameter showing the invester's skewness preference. Second, to have the Sharpe ratio with skewness for all investors, encouraged from the three-moment asset pricing model (Harvey and Siddique, 2000), we price second-order returns by projecting the second-order returns to first-order returns. The resulting ratio is the sum of the Sharpe ratio of first-order returns and the Sharpe ratio of projections of second-order returns. The Sharpe ratio with skewness for all investors have the same value of the skewness preference for all investors. Furthermore, The Sharpe ratio with skewness possesses properties of its increasing with return mean, decreasing with volatility, and non-decreasing with skewness. We use 299 market index to test the properties and confirm that the properties hold except for 9 indexes with a large negative skew, and a large kurtosis of returns[5].

Our research has both theoretical and practical contributions. The extended investment choice theory guides investors to choose a proper RAPM for their investments, and provides a way to discover a new RAPM. The Sharpe ratio with skewness can be used to accurately evaluate more assets with different return distributions than the Sharpe ratio. Also, it can be used in a portfolio construction and an investment decision. By using the Sharpe ratio with skewness, portfolio analytics will become more effective, portfolio efficiency will be improved, and right investment decisions will be made.

We divide the rest of the paper into five sections. In the second section, we extend the investment choice theory to include multiple risk measures. In the third section, we use the extended theory to infer the Sharpe ratio with skewness for an individual investor. In the fourth section, we derive the Sharpe ratio with skewness for all investors and verify its effectiveness using indexes' real return data. In the fifth section, we show three properties of the Sharpe ratio with skewness by mathematical derivation and an empirical work. We come to a conclusion in the last section.

2. The Investment Choice Theory

In this section, we extend the investment choice theory (Xiang *et al.*, 2012) to include more than one risk measure. We use the theory to derive a partial differential equation a RAPM must satisfy. We present a function form of a RAPM by solving the equation.

Investors usually make decisions before investing money in security markets. They first choose which markets and how much in a market they want to invest. They decide whether their investments have market indexes or interest rates (or their funding cost plus spreads) as investment benchmarks. Lastly, they determine how to measure performance and risk of their investments. They often use the information ratio as a RAPM if their investments are associated with market indexes while using the Sharpe ratio as a RAPM if their investments have interest rates as benchmarks. An investment choice includes all these decisions and leads to an investor's utility function.

We first give some notations and assumptions according to an investor's investment choice. Let a financial market or product be an investment set consisting of $n+1$ assets which are eligible for the investor to freely trade[6]. The investor wants to invest a portfolio formed by assets in the investment set and holds the portfolio over a discrete period. The investor expects that the ith asset has a return $E(r_i)$ in the period. The portfolio weights are w_i, $i=0,1,2,\cdots,n$, at the beginning of the period. The investor can be funded or unfunded according to the portfolio being 100 percent net long or not. In other words, the investor is funded when the weight constraint is $\sum_{i=0}^{n} w_i = 1$ and unfunded (or dollar neutral) when the weight constraint is $\sum_{i=0}^{n} w_i = 0$.

We also make the following assumptions for introducing a utility function and deriving a partial differential equation of a RAPM.

1) The investor cares only about the return and risks of the investment. It means that her expected utility function has only variables of expected return and some risk measures[7].
2) The portfolio has only one constraint: the weight constraint.
3) The investor maximizes her utility function for the give investment set and a given weight constraint.

There are two investment problems under the above assumptions. The problem for a funded investor is

[5] The 299 market indexes include asset classes such as equities, bonds, currencies, commodities, options, and hedge funds in different countries or regions such as Australia, China, German, Japan, South Africa, and Unite States.
[6] It means that an investor can buy or sell assets in the investment set without any transaction cost.
[7] The investor doesn’t change her utility function during an investment period.

Maximize: $E(U(r_p)) = f(u, v_1, \cdots, v_m)$

subject to $\begin{cases} u = \varphi(w_0, w_1, \cdots, w_n) \\ v_j = \psi_j(w_0, w_1, \cdots, w_n), j = 1, \cdots m \\ \sum_{i=0}^{n} w_i = 1 \end{cases}$

where f is the investor's utility function[8]. u and $v_j, j = 1, \cdots, m$ are corresponding to an expected return and risk measures and they are functions in the asset weights $w_i, i = 0,1,2,\cdots,n$. The shape of the utility function represents the investor's risk aversion towards her investment. Since the function is not identified, her risk aversion is unknown.

A Lagrangian of the maximization problem is

$$L = f(\varphi(w_0, \cdots, w_n), \psi_1(w_0, \cdots, w_n), \cdots, \psi_m(w_0, \cdots, w_n)) - \lambda\left(\sum_{i=0}^{n} w_i - 1\right),$$

where λ is the Lagrange multiplier. From the first order conditions for the Lagrangian, we obtain the investor's equilibrium condition:

$$\begin{cases} \frac{\partial L}{\partial w_i} = \frac{\partial f}{\partial u}\frac{\partial \varphi}{\partial w_i} + \sum_{j=1}^{m} \frac{\partial f}{\partial v_j}\frac{\partial \psi_j}{\partial w_i} - \lambda = 0, i = 0,1,\cdots,n \\ \frac{\partial L}{\partial \lambda} = \sum_{i=0}^{n} w_i - 1 = 0 \end{cases} \quad (2.1)$$

After subtracting the first equation from the next $n+1$ equations in (2.1), we have

$$\left(\frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial f}{\partial u} + \sum_{j=1}^{m}\left(\frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial f}{\partial v_j} = 0, i = 0,1,\cdots,n \quad (2.2)$$

Multiplying the i th equation in (2.2) by the weight w_i, and then summing all the resulting equations, we come to the following equation by the weight constraint $\sum_{i=0}^{n} w_i - 1 = 0$.

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial f}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial f}{\partial v_j} = 0 \quad (2.3)$$

Additionally, if the investor has the function $z = z(u, v_1 \cdots, v_m)$ as his RAPM for her investment, the value of the utility function should increase most as the value of the RAPM increases[9]. Consequently, the gradient of the function z must have the same direction as the gradient of the utility function f. Since the equation in (2.3) implies that the gradient of f

$$\nabla f = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v_1}, \cdots, \frac{\partial f}{\partial v_m}\right)^T$$

and the vector

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}, \sum_{i=0}^{n} w_i \frac{\partial \psi_1}{\partial w_i} - \frac{\partial \psi_1}{\partial w_0}, \cdots, \sum_{i=0}^{n} w_i \frac{\partial \psi_m}{\partial w_i} - \frac{\partial \psi_m}{\partial w_0}\right)^T$$

are orthogonal, where the symbol ^{T} is the transpose operator of a vector, the gradient of function z

$$\nabla z = \left(\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v_1}, \cdots, \frac{\partial z}{\partial v_m}\right)^T$$

[8] If we assume that the wealth at the beginning of a period is one, we have $w = 1 + r_p$, where w and r_p corresponds to the wealth and return at the end of the period. The expected utility can be a function of return, i.e., $E(U(w)) = E(U(1 + r_p))$.

[9] If this is not true, there is an ambiguous situation such as the value of the utility function doesn't increase while the value of the RAPM strictly increases, that violates the definition of a RAPM.

must be orthogonal to the vector. So we have:

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i} - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i} - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial z}{\partial v_j} = 0, \qquad (2.4)$$

which are the RAPM differential equation for a funded investor.

The asset r_0 plays a very important role when we derive the equation in (2.4). In practice, it is often the funding cost of an investor's portfolio. We would like to call the asset as a reference asset in the rest of this paper. Note that the reference asset must be in the investment set.

If an investor is unfunded, her investment problem is

$$\text{Maximize: } E(U(r_p)) = f(u, v_1, \cdots, v_m)$$

$$\text{subject to } \begin{cases} u = \varphi(w_0, w_1, \cdots, w_n) \\ v_j = \psi_j(w_0, w_1, \cdots, w_n),\ j = 1, \cdots m \\ \sum_{i=0}^{n} w_i = 0 \end{cases}$$

and the Lagrangian of the maximization problem is

$$L = f(\varphi(w_0, \cdots, w_n), \psi_1(w_0, \cdots, w_n), \cdots, \psi_m(w_0, \cdots, w_n)) - \lambda\left(\sum_{i=0}^{n} w_i\right).$$

The first order conditions for the Lagrangian is

$$\begin{cases} \dfrac{\partial L}{\partial w_i} = \dfrac{\partial f}{\partial u}\dfrac{\partial \varphi}{\partial w_i} + \sum_{j=1}^{m} \dfrac{\partial f}{\partial v_j}\dfrac{\partial \psi_j}{\partial w_i} - \lambda = 0, i = 0, 1, \cdots, n \\ \dfrac{\partial L}{\partial \lambda} = \sum_{i=0}^{n} w_i = 0 \end{cases}$$

We can obtain a RAPM equation for an unfunded investor as the case for the funded investor. The equation is

$$\left(\sum_{i=0}^{n} w_i \frac{\partial \varphi}{\partial w_i}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(\sum_{i=0}^{n} w_i \frac{\partial \psi_j}{\partial w_i}\right)\frac{\partial z}{\partial v_j} = 0 \qquad (2.5)$$

To find a solution to the equation in (2.4) or (2.5), we need conditions:

$$\begin{cases} \sum_{i=0}^{n} w_i \dfrac{\partial \varphi}{\partial w_i} = \varphi \\ \sum_{i=0}^{n} w_i \dfrac{\partial \psi_j}{\partial w_i} = \psi_j \end{cases} \qquad (2.6)$$

These conditions tell important properties of the expected return and risk measures. We will discuss the property later on. Under the conditions in (2.6), the equation in (2.4) turns out to be

$$\left(u - \frac{\partial \varphi}{\partial w_0}\right)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}\left(v_j - \frac{\partial \psi_j}{\partial w_0}\right)\frac{\partial z}{\partial v_j} = 0 \qquad (2.7)$$

and the equation in (2.5) becomes

$$u\frac{\partial z}{\partial u} + \sum_{j=1}^{m} v_j \frac{\partial z}{\partial v_j} = 0 \qquad (2.8)$$

Furthermore, if both partial derivatives $\frac{\partial \varphi}{\partial w_0}$ and $\frac{\partial \psi_j}{\partial w_0}$ are constant, $j = 1, \cdots, m$, i.e.,

$$\begin{cases} \dfrac{\partial \varphi}{\partial w_0} = a \\ \dfrac{\partial \psi_j}{\partial w_0} = b_j,\ j = 1, \cdots, m \end{cases} \qquad (2.9)$$

the equation in (2.7) become

$$(u - a)\frac{\partial z}{\partial u} + \sum_{j=1}^{m}(v_j - b_j)\frac{\partial z}{\partial v_j} = 0 \qquad (2.10)$$

According to the Euler's Lemma, the conditions in (2.6) are equivalent to the fact that the expected return φ and risk measures $\psi_j, j=1,\cdots,m$ are homogeneous functions of degree one. The homogeneity means

$$\varphi(c_0 w_0,\cdots,c_0 w_n)=c_0\varphi(w_0,\cdots,w_n) \tag{2.11}$$

and

$$\psi_j(c_1 w_0,\cdots,c_1 w_n)=c_1\psi_j(w_0,\cdots,w_n),\ j=1,\cdots,m \tag{2.12}$$

where c_0 and c_1 are positive constants. The conditions in (2.11) and (2.12) inform us that the expected return and risk measures should increase or decrease as the same rate as the asset weights. The condition in (2.12) is consistent with the positive homogeneity axiom of a coherent risk measure defined by Artzner *et al.* (1999).

Since the equation in (2.10) is a Lagrange's equation, it is easy to get their general solution by solving the equations:

$$\frac{du}{u-a}=\frac{dv_1}{v_1-b_1}=\cdots=\frac{dv_m}{v_m-b_m}=\frac{dz}{0} \tag{2.13}$$

The equations in (2.13) have the solution:

$$\begin{cases} \dfrac{u-a}{v_1-b_1}=C_1 \\ \dfrac{v_j-b_j}{v_1-b_1}=C_j,\ j=2,\cdots,m \\ z=C \end{cases}$$

where $C_j, j=1,\cdots,m$ and C are constants. As a result, the general solution to the equation in (2.10) is

$$z=k\left(\frac{u-a}{v_1-b_1},\frac{v_2-b_2}{v_1-b_1},\cdots,\frac{v_m-b_m}{v_1-b_1}\right) \tag{2.14}$$

where k is a function in m variables. Therefore, a funded investor's RAPM is given by the function form in (2.14). Similarly, an unfunded investor's RAPM has the function form:

$$z=k\left(\frac{u}{v_1},\frac{v_2}{v_1},\cdots,\frac{v_m}{v_1}\right) \tag{2.15}$$

We conclude in this section that a RAPM must satisfy the equation in (2.8) or (2.10) if an investor has the expected utility: $E(U(r_p))=f(u,v_1,\cdots,v_m)$. The RAPM is known by the function form in (2.14) for a funded investor under the conditions in (2.9), (2.11) and (2.12) and the function form in (2.15) for an unfunded investor under the conditions in (2.11) and (2.12).

3. The Sharpe Ratio with Skewness for an Individual

We use the investmentchoice RAPM theory to infer the Sharpe ratio with skewness. The Sharpe ratio with skewness is the original Sharpe ratio adjusted by return skewness[10]. Particularly, it is the original Sharpe ratio when returns has no skewness. In this section, we first derive the Sharpe ratio with skewness for a funded investor, then the one for an unfunded investor.

Suppose that a funded investor has the expected utility

$$E(U(r_p))=f(E(r_p),\sigma(r_p),\gamma(r_p)) \tag{3.1}$$

where $\gamma(r_p)=\left[E(r_p-E(r_p))^3\right]^{\frac{1}{3}}$ is the return skewness, and the reference asset r_0 is a risk-free asset r_f. In the extended theory, we can assume that

$u=\varphi(w_0,w_1,\cdots,w_n)=E(r_p)$,

[10] We call the Sharpe ratio defined by as the original Sharpe ratio to distinguish from the Sharpe ratio with skewness.

$$v_1=\psi_1(w_0,w_1,\cdots,w_n)=\sigma(r_p),$$

and

$$v_2=\psi_2(w_0,w_1,\cdots,w_n)=\gamma(r_p).$$

Since

$$\frac{\partial\varphi}{\partial w_0}=\frac{\partial(E(r_p))}{\partial w_0}=\frac{\partial\left(\sum_{i=0}^{n}w_iE(r_i)\right)}{\partial w_0}=E(r_0)=r_f,$$

$$\frac{\partial\psi_1}{\partial w_0}=\frac{\partial\sigma(r_p)}{\partial w_0}=\frac{\partial\sigma\left(\sum_{i=0}^{n}w_ir_i\right)}{\partial w_0}=\frac{\partial\sigma\left(\sum_{i=1}^{n}w_ir_i\right)}{\partial w_0}=0,$$

and

$$\frac{\partial\psi_2}{\partial w_0}=\frac{\partial\gamma(r_p)}{\partial w_0}=\frac{\partial\gamma\left(\sum_{i=0}^{n}w_ir_i\right)}{\partial w_0}=\frac{\partial\gamma\left(\sum_{i=1}^{n}w_ir_i\right)}{\partial w_0}=0,$$

the conditions in (2.9) hold. On the other hand, we have

$$\varphi(c_0w_0,\cdots,c_0w_n)=E(c_0r_p)=c_0E(r_p)=c_0\varphi(w_0,\cdots,w_n),$$

$$\psi_1(c_1w_0,\cdots,c_1w_n)=\sigma(c_1r_p)=c_1\sigma(r_p)=c_1\psi_1(w_0,\cdots,w_n),$$

and

$$\psi_2(c_1w_0,\cdots,c_1w_n)=\gamma(c_1r_p)=c_1\gamma(r_p)=c_1\psi_2(w_0,\cdots,w_n),$$

where c_0 and c_1 are positive constants. The conditions in (2.11) and (2.12) are satisfied. By (2.14), the funded investor has the following RAPM:

$$z=k\left(\frac{E(r_p)-r_f}{\sigma(r_p)},\frac{\gamma(r_p)}{\sigma(r_p)}\right) \tag{3.2}$$

Obviously, the RAPM consists of the first three central moments of return.

To order to find a simple RAPM for the funded investor, we consider this kind of the function k, $k(x,y)=k_1(x)k_2(y)$, where k_1 and k_2 are two functions in single variable. By (3.2), the RAPM is

$$z=k_1\left(\frac{E(r_p)-r_f}{\sigma(r_p)}\right)k_2\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right).$$

In view of the fact that the RAPM is the original Sharpe ratio when returns do not show any skewness, we have

$$k_1\left(\frac{E(r_p)-r_f}{\sigma(r_p)}\right)=\frac{E(r_p)-r_f}{\sigma(r_p)}$$

and $k_2(0)=1$.

As the volatility $\sigma(r_p)$ is the squared root of the second central moment and the skewness $\gamma(r_p)$ is the cubed root of the third central moment, let $k_2(y)=a+by+cy^2+dy^3$ by approximating k_2 with a 3rd degree Taylor polynomial, where a, b, c and d are constants. It follows immediately from $k_2(0)=1$ that $a=1$. The RAPM becomes

$$z=\frac{E(r_p)-r_f}{\sigma(r_p)}\left(1+b\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)+c\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2+d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3\right) \tag{3.3}$$

If an investor prefers a positive return skewness to a negative return skewness, the RAPM must satisfy $\partial z/\partial\gamma(r_p)\geq 0$, i.e.,

$$\frac{\partial z}{\partial\gamma(r_p)}=\frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(b+2c\frac{\gamma(r_p)}{\sigma(r_p)}+3d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2\right)\geq 0.$$

Since the skewness $\gamma(r_p)$ can be positive or negative, we have $c=0$. So

$$\frac{\partial z}{\partial \gamma(r_p)} \equiv \frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(b+3d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^2\right),$$

where d must be a non-negative constant. Furthermore, we can write the RAPM in (3.3) as

$$z=\frac{E(r_p)-r_f}{\sigma(r_p)}\left(1-de^3+d\left(\frac{\gamma(r_p)}{\sigma(r_p)}+e\right)^3\right)$$

for some constant *e*. Since an investor prefers a RAPM with a simple form than a complicated form, it can be assumed that the constant *e* be zero. As a result, the RAPM for a funded investor becomes

$$z=\frac{E(r_p)-r_f}{\sigma(r_p)}\left(1+d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3\right). \qquad (3.4)$$

The parameter d is a gauge how an investor prefers return skewness. Its non-negativity indicates that an investor like investment returns with positive skewness than negative skewness when other things are equal. An investor can assign a small (large) value to the parameter in her RAPM if believing return skewness has a small (large) impact on her portfolio. The parameter represents an investor's skewness preference.

Similarly, we can infer a RAPM with skewness for an unfunded investor. We skip the details for the inference. The RAPM for an unfunded investor is

$$z=\frac{E(r_p)}{\sigma(r_p)}\left(1+d\left(\frac{\gamma(r_p)}{\sigma(r_p)}\right)^3\right) \qquad (3.5)$$

We call the RAPM in (3.4) or (3.5) as the Sharpe ratio with skewness.

4. The Sharpe Ratio with Skewness for all Investors

The Sharpe ratio with skewness in (3.4) or (3.5) is associated with the skewness preference of an investor, i.e., the parameter *d*. In this section, we will introduce the Sharpe ratio with skewness for all investor from another perspective - an asset pricing model. The Sharpe ratio with skewness will give an explicit expression to the parameter.

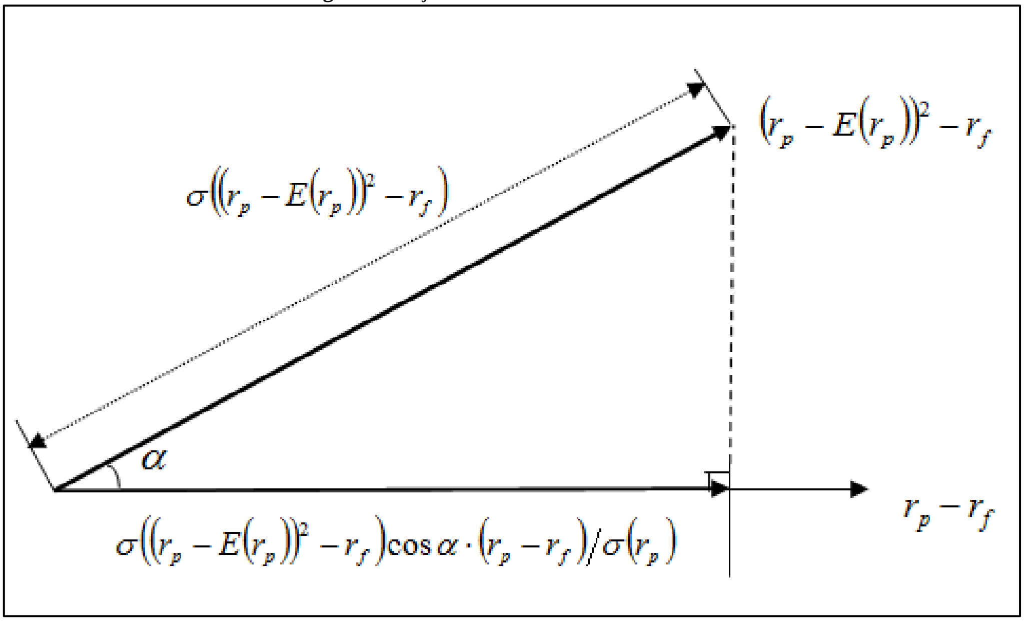
If an investor prefers a positive skew to a negative skew in asset returns, return skewness must have a price on it. Harvey and Siddique (2000), present a pricing model among other three-moment asset pricing models (Friend and Westerfield, 1980; Ingersoll, 1990; Kraus and Litzenberger, 1976; Lim, 1989; Nummelin, 1994; Sears and Wei, 1985; Waldron, 1990). They develop their model by assuming the pricing kernel is a quadratic function in the return of a market portfolio. The model reminds us that return skewness is closely related to second-order returns. Thus, the Sharpe ratio with skewness would be the sum of the Sharpe ratio with first-order returns (the original Sharpe ratio) and the Sharpe ratio with second-order returns.

Figure 1 shows how we price return skewness using second-order returns. In the figure,

$$\cos\alpha=\rho\left(r_p-r_f,(r_p-E(r_p))^2-r_f\right) \qquad (4.1)$$

where α is the angle between the first-order return, r_p-r_f, and the second-order return, $(r_p-E(r_p))^2-r_f$. We project the second-order return onto the first-order return.

Figure-1. Projection of the Second-order Return



The price of skewness is the projection of the second-order return along the first-order return, that is

$$\sigma((r_p - E(r_p))^2 - r_f)\cos\alpha \frac{E(r_p) - r_f}{\sigma(r_p)}$$

$$= \sigma((r_p - E(r_p))^2 - r_f)\rho(r_p - r_f, (r_p - E(r_p))^2 - r_f)\frac{E(r_p) - r_f}{\sigma(r_p)}.$$

As the total risk of the projection is $\sigma((r_p - E(r_p))^2 - r_f)$, the Sharpe ratio due to the second-order returns is

$$\cos\alpha \cdot \frac{E(r_p) - r_f}{\sigma(r_p)} = \rho(r_p - r_f, (r_p - E(r_p))^2 - r_f)\frac{E(r_p) - r_f}{\sigma(r_p)} \tag{4.2}$$

Next, we simplify the correlation: $\rho(r_p - r_f, (r_p - E(r_p))^2 - r_f)$. Let's denote $\theta(r_p) = E([r_p - E(r_p)]^4)^{1/4}$ as the return kurtosis[11]. Since

$$\rho(r_p - r_f, (r_p - E(r_p))^2 - r_f) = \frac{Cov(r_p - r_f, (r_p - E(r_p))^2 - r_f)}{\sigma(r_p)\sigma((r_p - E(r_p))^2 - r_f)} = \frac{Cov(r_p, (r_p - E(r_p))^2)}{\sigma(r_p)\sigma((r_p - E(r_p))^2)}$$

and

$$\sigma((r_p - E(r_p))^2) = \sqrt{E((r_p - E(r_p))^4) - (E(r_p - E(r_p))^2)^2} = \sigma(r_p)\sqrt{\theta(r_p)^4 - \sigma(r_p)^4},$$

we have

$$\rho(r_p - r_f, (r_p - E(r_p))^2 - r_f)$$

$$= \frac{\sigma((r_p - E(r_p))^3)}{\sigma(r_p)\sigma\sqrt{\theta(r_p)^4 - \sigma(r_p)^4}}$$

$$= \frac{\gamma(r_p)^3}{\sigma(r_p)\sigma\sqrt{\theta(r_p)^4 - \sigma(r_p)^4}}$$

$$= \frac{(\gamma(r_p)/\sigma(r_p))^3}{\sqrt{(\theta(r_p)/\sigma(r_p))^4 - 1}}.$$

[11] We call $[E(r_p - E(r_p))]^4/\sigma(r_p)^4$ as the normalized kurtosis in this paper, instead of as the kurtosis defined by most authors.

By (4.2), the Sharpe ratio due to the second-order returns is

$$\cos\alpha \cdot \frac{E(r_p)-r_f}{\sigma(r_p)} = \frac{(\gamma(r_p)/\sigma(r_p))^3}{\sqrt{(\theta(r_p)/\sigma(r_p))^4-1}} \frac{E(r_p)-r_f}{\sigma(r_p)}. \tag{4.3}$$

After combining the original Sharpe ratios and the Sharpe ratio due to the second-order returns in (4.3), we obtain the Sharpe ratio with skewness as

$$z = \frac{E(r_p-r_f)}{\sigma(r_p)}(1+\cos\alpha) = \frac{E(r_p-r_f)}{\sigma(r_p)}\left(1+\frac{(\gamma(r_p)/\sigma(r_p))^3}{\sqrt{(\theta(r_p)/\sigma(r_p))^4-1}}\right) \tag{4.4}$$

The Sharpe ratio with skewness in (4.4) is the original Sharpe ratio adjusted by normalized skewness and normalized kurtosis. More precisely, it is the original Sharpe ratio multiplied by the amount: $1+\cos\alpha$. The Sharpe ratio with skewness is adjusted upward (downward) when the angle α is acute (obtuse). An acute (obtuse) angle α implies that return skewness $\gamma(r_p)$ is positive (negative). Particularly, it is the original Sharpe ratio when the angle α between the first-return and second-order return is perpendicular , i.e., the skewness $\gamma(r_p)$ is zero. Please note that the angle α is closely related to the relative position of the second-order return to the first-order return.

The Sharpe ratio with skewness in (4.4) is the same as the one in (3.4) if the skewness preference d is equal to

$$d = \frac{1}{\sqrt{(\theta(r_p)/\sigma(r_p))^4-1}} \tag{4.5}$$

The skewness preference in (4.5) turns into smaller (larger) as the normalized kurtosis becomes large (small). This means that investors like large return kurtosis when returns show negative skewness, and small return kurtosis when returns show positive skewness.

Table 1 shows return statistics of 20 market indexes. The statistics includes index number, return mean, volatility, skewness, kurtosis, and indexes' values of $1 at the end of a period. The last two columns are the original Sharpe ratios and Sharpe ratios with skewness of indexes. The Sharpe ratios are calculated by assuming the risk-free rates are zero. The average of the original Sharpe ratio is 0.68 and the average of the Sharpe ratio with skewness is 0.65. The original Sharpe ratios range from 0.32 to 0.99 while the Sharpe ratios with skewness range from 0.25 to 1.04. The former range is narrower than the later range.

Table-1. Return Statistics of Market Indexes

Index Number	Return Mean (%)	Volatility (%)	Skewness	Kurtosis	End Value of $1	Original Sharpe ratio	Sharpe Ratio with Skewness
1	14.85	16.87	0.42	9.56	60.02	0.88	1.01
2	15.04	15.63	0.10	5.71	70.79	0.96	1.01
3	14.74	15.40	0.18	5.25	63.26	0.96	1.04
4	13.87	14.74	0.05	5.54	45.05	0.94	0.96
5	14.22	14.36	-0.09	5.21	53.73	0.99	0.95
6	12.44	14.94	-0.17	4.01	23.83	0.83	0.75
7	12.78	15.36	-0.07	3.76	26.93	0.83	0.80
8	11.86	15.55	-0.25	5.19	17.68	0.76	0.67
9	11.91	15.52	-0.21	4.43	18.16	0.77	0.68
10	9.01	15.08	-0.26	4.60	5.27	0.60	0.52
11	10.13	16.71	-0.13	5.40	7.67	0.61	0.57
12	11.04	17.30	-0.21	4.75	10.93	0.64	0.57
13	11.37	17.51	-0.28	5.10	12.37	0.65	0.56
14	9.40	18.15	0.03	5.66	5.00	0.52	0.53
15	9.14	18.88	-0.34	5.83	4.17	0.48	0.41
16	9.54	18.18	0.05	4.02	5.33	0.52	0.54
17	6.29	19.39	-0.42	4.51	1.14	0.32	0.25
18	9.38	19.73	-0.19	5.16	4.34	0.48	0.43
19	8.17	21.93	-0.30	4.98	2.07	0.37	0.32
20	11.80	24.91	-0.31	4.80	7.39	0.47	0.40
Mean	11.35	17.31	-0.12	5.17	22.26	0.68	0.65
Min	6.29	14.36	-0.42	3.76	1.14	0.32	0.25
Max	15.04	24.91	0.42	9.56	70.79	0.99	1.04

Data Source: FactSet

We compare the original Sharpe ratio with the Sharpe ratio with skewness to see which ratio does a better job to explain end values of the 20 market indexes. Based on our calculation, the correlation between the original Sharpe ratios and end values of the indexes is 0.89. The correlation between the Sharpe ratios with skewness and end values of the indexes is 0.94.The correlations indicate that the Sharpe ratio with skewness better explain end values of the market indexes than the original Sharpe ratio.

5. Robustness

Investors undoubtedly like an investment with positive expected return, low volatility, and /or positive skewness. The preferences require that the Sharpe ratio with skewness must possesses three properties: it increases with expected return, decreases with volatility, and non-decreases with skewness. This section will mathematically and empirically verify the properties of the Sharpe ratio with skewness.

We can mathematically describe the three properties by three partial derivative inequalities: $\partial z/\partial E(r_p)>0$, $\partial z/\partial \sigma(r_p)<0$, and $\partial z/\partial \gamma(r_p)\geq 0$. For a calculation purpose, we rewrite the Sharpe ratio with skewness in (4.4) as

$$z=\frac{E(r_p-r_f)}{\sigma(r_p)}(1+\cos\alpha)=\frac{E(r_p-r_f)}{\sigma(r_p)}\left(1+\frac{\gamma(r_p)^3}{\sigma(r_p)\sqrt{\theta(r_p)^4-\sigma(r_p)^4}}\right) \quad (5.1)$$

We assume that $E(r_p)-r_f$ is non-negative and $1+\cos\alpha$ is positive. From (5.1), we compute the partial derivatives of z with respect to the expected return and skewness and get

$$\frac{\partial z}{\partial E(r_p)}=\frac{1}{\sigma(r_p)}(1+\cos\alpha)>0$$

and

$$\frac{\partial z}{\partial \gamma(r_p)}=\frac{E(r_p)-r_f}{\sigma(r_p)}\left(\frac{3\gamma(r_p)^2}{\sigma(r_p)\sqrt{\theta(r_p)^4-\sigma(r_p)^4}}\right)\geq 0.$$

Thus, the Sharpe ratio with skewness satisfies the first property and the third property.

Now we check that the Sharpe ratio with skewness satisfies the second property. First, the Sharpe ratio with skewness has the following partial derivative with respective to volatility:

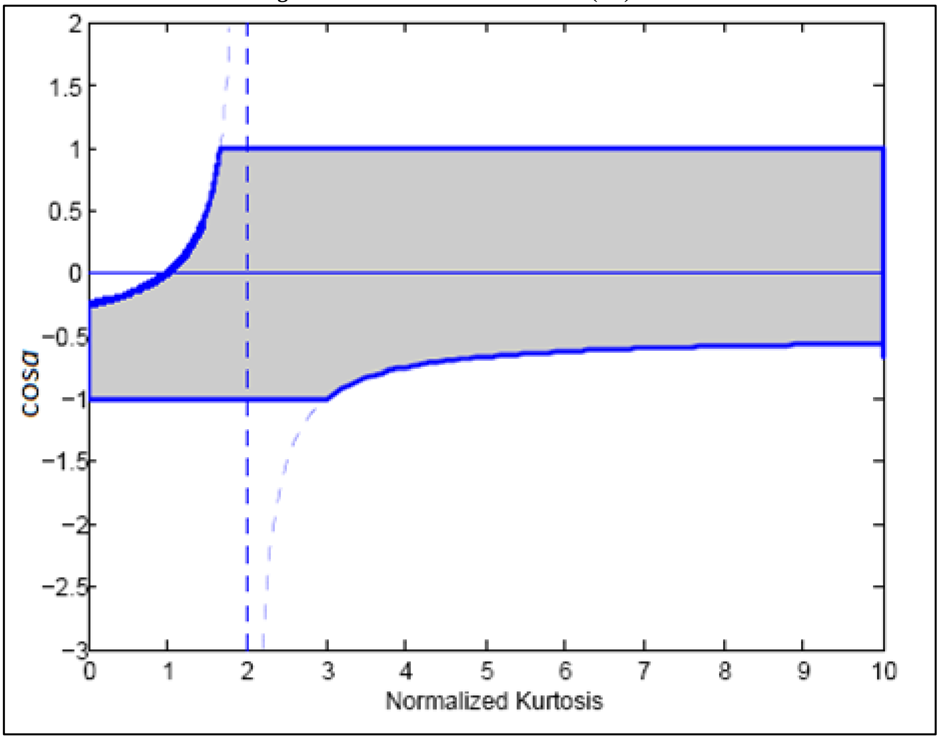
$$\begin{aligned}\frac{\partial z}{\partial \sigma(r_p)}&=\frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-\frac{2\gamma(r_p)^3\left(\theta(r_p)^4-2\sigma(r_p)^4\right)}{\sigma(r_p)\left(\theta(r_p)^4-\sigma(r_p)^4\right)^{\frac{3}{2}}}\right)\\&=\frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-\frac{2\left(\theta(r_p)^4-2\sigma(r_p)^4\right)}{\theta(r_p)^4-\sigma(r_p)^4}\frac{\gamma(r_p)^3}{\sigma(r_p)\sqrt{\theta(r_p)^4-\sigma(r_p)^4}}\right)\\&=\frac{E(r_p)-r_f}{\sigma(r_p)^2}\left(-1-2\frac{\left(\theta(r_p)/\sigma(r_p)\right)^4-2}{\left(\theta(r_p)/\sigma(r_p)\right)^4-1}\cos\alpha\right).\end{aligned}$$

The Sharpe ratio with skewness has the second property if

$$-2\frac{\left(\theta(r_p)/\sigma(r_p)\right)^4-2}{\left(\theta(r_p)/\sigma(r_p)\right)^4-1}\cos\alpha<1 \quad (5.2)$$

Figure 2 shows the area where the condition in (5.2) holds. Specially, the condition is true when $3/2\leq\left(\theta(r_p)/\sigma(r_p)\right)^4\leq 2$ and $\cos\alpha<1/2$, or $2<\left(\theta(r_p)/\sigma(r_p)\right)^4$ and $\cos\alpha\geq -1/2$.

Figure-2. Area where the condition in (5.2) holds



Please note that if $E(r_p) - r_f$ is not non-negative, i.e., strictly negative, the Sharpe ratio with skewness will increase with volatility and decrease with skewness.

We test the condition in (5.2) using return statistics of 299 market indexes. The indexes include asset classes such as equities, bonds, currencies, commodities, options, and hedge funds in different countries or regions around the world. The statistics are from FactSet[12]. The test shows that 290 indexes satisfy the condition and 9 indexes violate the condition. The 9 indexes have a large return skewenss and kurtosis. 6 of them are hedged funds with a very small volatility[13]. From the test, we confidently believe that the Sharpe ratio with skewness can accurately evaluate more assets with different return attributions than the original Sharpe ratio.

6. Conclusion

This paper extends the investment choice theory to overcome improperly or inaccurately use of a risk-adjusted performance measure. It involves four stages. First, we determine an expected utility function form an investor's investment choice. Second, we draw from a partial differential equation by maximizing the utility function. Third, we show that a risk-adjusted performance measure must satisfy the equation. Finally, we use the theory to infer the Sharpe ratio with skewness for an individual investor. With this theory, investors anchor a risk-adjusted performance measure to their investment choice so they can choose a proper risk-adjusted performance measure and discover a new risk-adjusted performance measure for their investments.

From another perspective, we derive the Sharpe ratio with skewness for all investors by pricing second-order returns using first-order returns. In this derivation, the resulting ratio is the Sharpe ratio due to the first-order return plus the Sharpe ratio due to the second-order return. We show that the Sharpe ratio with skewness increases with return mean, decreases with volatility, and non-decreases with skewness. Tests show that the new performance measure can accurately evaluate more assets than the Sharpe ratio does. Potentially improving evaluation and performance of investments, the proposed Sharpe ratio with skewness contributes to portfolio analytics, portfolio construction, and investment decisions given that a risk-adjusted performance measure is a necessary gauge in an investment process.

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[12] The data are available upon request.
[13] There are 51 hedge fund indexes among the 299 market indexes.

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