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Fuzzy Logic: History, Methodology and Applications to Education

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Abstract

Fuzzy Logic has been evolved today to a valuable extension and necessary supplement of the traditional bi-valued Logic of Aristotle, with applications covering almost all the specter of human activities. This new logic of infinite values, developed rapidly during the last 50 years, is based on the notion of fuzzy set introduced by Zadeh in 1965. The target of the present review article is twofold: First to give to the non expert a general idea about the content and the perspectives of Fuzzy Logic. Second to present applications of it to Education (student assessment), which constitute part of the author's research work during the last twenty years on building fuzzy models representing several real life situations. Thus, the article includes a brief account of the history and development of Fuzzy Logic, the ways of dealing with the uncertainty in a fuzzy environment, the use of the Centre of Gravity (COG) defuzzification technique as an assessment method, as well as the triangular fuzzy numbers (TFNs) and their arithmetic. Note that the COG technique is the most popular defuzzification method used in fuzzy mathematics, whereas the TFNs is the simplest form of fuzzy numbers, which play in general an important role in fuzzy mathematics, analogous to the role of the ordinary numbers for the traditional mathematics.

Keywords: Fuzzy set (FS); Fuzzy logic (FL); Uncertainty; Centre of gravity (COG) defuzzification technique; Triangular fuzzy mumbers (TFNs); Student assessment.

1. Introduction

A few years ago Probability theory used to be the unique tool in hands of the experts for dealing with situations of uncertainty appearing in problems of all sciences and of our everyday life. However nowadays, with the development of Fuzzy Set (FS) theory and in extension of Fuzzy Logic (FL), things have been changed. In fact, these new

mathematical tools gave to the scientists the opportunity to model under conditions which are not precisely defined and as a result the spectre of their applications has been rapidly extended covering, apart from the Physical Sciences, sectors like Economics and Management, Industry, Robotics, Decision Making, Programming, Fuzzy Control, Medicine, Biology, Humanities, Education and almost all the other sectors of human activity including human reasoning itself (Klir and Folger, 1988; Voskoglou, 2017). Expert systems, like financial planners, diagnostic, meteorological, information -retrieval, control systems, etc, have been the most obvious recipients of the benefits of FL (Umbers and King, 1980), since their domain is often characterized by vagueness. The first major commercial application of FL was in cement kiln control (Zadel, 1983), followed by a navigation system for automatic cars, a fuzzy controller for automatic operation of trains, laboratory level controllers, controllers for robot vision, graphics controllers for automated police sketchers and many others. It must be mentioned that fuzzy mathematics has been also significantly developed in theoretical level providing important contributions even in branches of the classical mathematics, like Algebra, Analysis, Geometry, etc.

The target of this work is to give to the non expert on the subject a general idea about the content and the perspectives of FL and to present applications connected to assessment problems aiming to show the usefulness of FL for Education. The rest of the article is formulated as follows: The history and development of FL is briefly exposed in the second Section. The third Section deals with the ways of manipulating the uncertainty in fuzzy systems. The Centre of Gravity (COG) technique, the most popular defuzzification method of fuzzy mathematics, is described in the next Section. The Triangular Fuzzy Numbers and their

Arithmetic are presented in the fifth Section. The article closes with the final conclusion stated in the sixth Section.

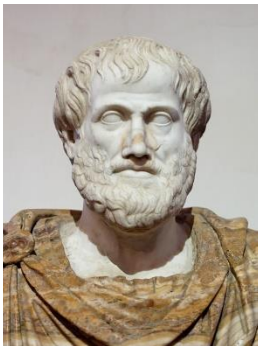
2. History of Fuzzy Logic

Human Reasoning was dominated for centuries by the fundamental "Laws of Thought" (Korner, 1967), introduced by Aristotle (384-322 BC) and the philosophers that preceded him, which include:

- The principle of identity
- The law of the excluded middle
- The law of contradiction

In particular, the second of the above laws, stating that every proposition has to be either "True" or "False", was the basis for the genesis of the Aristotle's bi-valued Logic. The precision of the traditional mathematics owes undoubtedly a large part of its success to this Logic.

Picture-1. Roman copy in marble of a Greek bronze bust of Aristotle by Lysippus, 330 BC



However, even when Parmenides proposed, around 400 BC, the first version of the law of the excluded middle, there were strong and immediate objections. For example, Heraclitus opposed that things could be simultaneously true and not true, whereas the Buddha Sidhartha Gautama, who lived in India a century earlier, had already indicated that almost every notion contains elements from its opposite one. The ancient Greek philosopher Plato (427-377 BC) laid the foundation of what it was later called FL by claiming that there exists a third area beyond "True" and "False", where these two opposite notions can exist together. More modern philosophers like Hegel, Marx, Engels and others adopted and further cultivated the above Plato's belief.

The Polish philosopher Jan Lukasiewicz (1878-1956) was the first to propose a systematic alternative of the bi-valued logic introducing in the early 1900's a three valued logic by adding the term "Possible" between "True" and "False" (Lejewski, 1967). Eventually he developed an entire notation and axiomatic system from which he hoped to derive modern mathematics. Later he also proposed four and five valued Logics and he finally arrived to the conclusion that axiomatically nothing could prevent the derivation of an infinite valued Logic.

But it was not until relatively recently that an infinite-valued Logic was introduced (Zadeh, 1973), called FL, because it is based on the notion of FS initiated in 1965 (Zadeh, 1965) by Lotfi (2018), Professor at the University of Berkeley, California. An important goal of FL is that through it algorithmic procedures can be devised which translate the "fuzzy" terminology into numerical values, perform reliable operations upon those values and then return natural language statements in a reliable manner.

Zadeh (1921–2017) (Wikipedia, retrieved from the Web on February, 2012) was born in Baku, Azerbaijan of USSR, to a Russian Jewish mother (Fanya Koriman), who was a pediatrician, and an Iranian Azeri father (Rahim Aleskerzade), who was a journalist on assignment from Iran.

Picture-2. L.A. Zadeh (1921–2017)

At the age of 10, when Stalin introduced collectivization of farms in USSR, the Zadeh family moved to Iran. In 1942 Zadeh graduated from the University of Tehran with a degree in electrical engineering and moved to the USA in 1944. He received a MS from MIT in 1946 and a Ph.D. in electrical engineering from Columbia University in 1949. He taught for ten years in Columbia, promoted to a Full Professor in 1957, before moving to Berkeley in 1959. Among others he introduced jointly with J.R. Ragazzini in 1962 the pioneering *z-transform method* used today in the digital analysis (Brule, 2016) whereas his more recent works include computing with words and perceptions (Zadeh, 1984;2005a) and an outline towards a generalized theory of uncertainty (Zadeh, 2005b). It has been estimated that Zadeh, who died in Berkeley on 6 September 2017, aged 96, counted in 2011 more than 950 000 citations by other researchers!

As it was expected, the far-reaching theory of fuzzy systems aroused some objections to the scientific community. While there have been generic complaints about the fuzziness of assigning values to linguistic terms, the most cogent criticisms come from Haak (1979). She argued that there are only two areas – the nature of Truth and Falsity and the fuzzy systems' utility – in which FL could be possibly needed, and then maintained that in both cases it can be shown that FL is unnecessary.

Fox (1981) responded to her objections indicating that FL is useful in three areas: To handle real-world relationships which are inherently fuzzy, to calculate the frequently existing in real world situations fuzzy data and to describe the operation of some inferential systems which are inherently fuzzy. His most powerful arguments were that traditional and FL need not be seen as competitive, but as complementary and that FL, despite the objections of classical logicians, has found its way into practical applications and has proved very successful there.

3. The concept of Fuzzy Set

Real life situations appear frequently where some definitions have not clear boundaries, like "the young people of a city", "the good players of a team", "the diligent students of a class", etc. The need to model mathematically such kind of situations was one of the main reasons that laid to the development of the FS theory.

Let *U* be the set of the discourse. Then, according to Zadeh (1965), a *fuzzy subset A of U* (or for brevity a FS in *U*) can be defined with the help of its *membership function*

m_A: $U \to [0,1]$, which assigns to each element *x* of *U* a real value $m_A(Fox)$ in [0, 1], called the *membership degree of x in A*. The closer is $m_A(Fox)$ to 1, the more *x* satisfies the characteristic property of *A*. Then one defines *A* as a set of ordered pairs of the form: $A = \{(x, m_A(x)) : x \in U\}$

Many authors, for reasons of simplicity, identify the FS *A* with its membership function m_A. A FS can be also denoted in the form of a symbolic sum, or a symbolic power series, or a symbolic integral, when *U* is a finite or numerable set or it has the power of the continuous respectively. For general facts on FS and the uncertainty connected to them we refer to the book of Klir and Folger (1988).

Example 1: The young human ages

Let *U* be the set of the non negative integers not exceeding 140 (considered as the upper bound of human life) representing the human ages. The set of all ages not exceeding a given integer in *U*, e.g. 20, is a crisp subset of *U*. On the contrary the set *A* of the young human ages, being not precisely defined, is a FS in *U*. The membership function of *A* can be defined by $m_A(x) = \begin{cases} \left[1+(0.04x)^2\right]^{-1}, & x \le 40 \\ 0, & x > 40 \end{cases}$

Therefore, the age of a recently born baby has membership degree $m_A(0) = 1$, the age of 25 years has membership degree $m_A(25) = (1+1^2)^{-1} = 0.5$, etc.

Remarks: 1) The membership function of a FS is not uniquely determined, its definition depending on the user's personal criteria, which are based on statistical, empirical or logical observations. Nevertheless the creditability of a FS for modeling a vague real situation depends on the successful or not definition of its membership function.

2) A crisp subset *B* of *U* can be considered as a FS in *U* with membership function $m_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$. On the basis of this consideration most of the notions and operations concerning the crisp sets, like subset, complement, union, intersection, Cartesian product, binary and other relations, etc., can be extended to FS. Moreover, the Zadeh's *extension principle* (Klir and Folger, 1988) enables the transform of models with crisp variables to corresponding models with fuzzy variables. Therefore, the argument that FL is a natural extension of the bi-valued logic is strongly justified by those facts.

3) It must be emphasized that, despite to the fact that FS and Probability theories act on the same real interval [0, 1], they *essentially differ* to each other. For example, the expression "The probability that John is tall is 85%" has a completely different meaning from the FL statement "John's membership degree in the FS of the tall men is 0.85". In fact, the former expression, in terms of the law of the excluded middle, means that John, being an unknown to the observer person, is either tall or short, but according to its origin the probability to be tall is 85%. On the contrary, the latter statement, since John's membership degree in the FS of the tall men approaches 1, means that John is more or less a tall person.

4) There are also other differences between the above two theories mainly arising from the way of defining the corresponding notions and operations. For instance, whereas the sum of the probabilities of all the single events (singleton subsets) of *U* is always 1 (probability of the certain event), this is not necessarily true for the membership degrees. Consequently, whereas a probability distribution could be used to define membership degrees, the converse is not always true.

4. Managing the uncertainty of fuzzy systems

A system's *uncertainty* can be roughly considered as being the lack of exact knowledge and/or information of the data describing it. Two categories of uncertainty emerge quite naturally, captured by the terms *vagueness* and *ambiguity* respectively. Namely, some domain of interest is said to be vague, if it can not be delimited by sharp boundaries. Ambiguity on the other hand is associated with situations in which the choice between two or more alternatives is left unspecified.

According to the fundamental principle of the classical Information theory, the quantity of information produced by an activity taking place within a system can be measured by the resulting reduction of the system's uncertainty. Shannon (1948), based on the above principle and on Hartley (1928) earlier work, obtained using probabilities the mathematical definition of *information*, which appears to be analogous to the definition of the well known from Physics *entropy* of a physical system. For this reason, the Shannon's formula calculating the information in terms of the associated to it uncertainty is better known as the *Shannon's entropy*.

As it is mentioned in p. 20 of a paper due to Klir (1995), when the universal set U is a finite set, Shannon's formula can be adapted for use in a fuzzy environment to the form H = $-\sum_{s=1}^{n} m_s \log_a m_s$, where m: $U \to [0, 1]$ is the membership function of the FS describing the existing uncertainty, $m_s = m(s)$ denotes the membership degree of the element s of U in this FS and n is the cardinality of U. Usually in practical applications the basis α of the logarithms takes one of the values 2, 10 or e (the basis of the natural logarithms). Accordingly the resulting information is measured in *bits, nats* or *dits,* which are acronyms of the expressions "(Subbotin *et al.*)nary digi(van Broekhoven and Debaets)", "(na)tural digi(van Broekhoven and Debaets)" and "(d)ecimal dig(its)" respectively.

Here the Shannon's entropy will be used in the form H = $-\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s$ (1), where the sum is divided by the natural logarithm of n in order to be normalized. Thus H takes values in the interval [0, 1].

Example 2: Let U = {A, B, C, D, F} be the set of the linguistic grades A = Excellent, B = Very good, C = Good, D = Fair (passed) and F= Unsatisfactory (failed). Calculate the existing uncertainty in a class T of 60 students, who obtained the following grades in their mathematics exam: A = 5 students, B = 12, C = 20,

D = 18 and F = 5 students.

Solution: The student class can be represented as a FS in U in the form $T = \{(A, \frac{5}{60}), (B, \frac{12}{60}), (C, \frac{20}{60}), (D, \frac{18}{60}), (F, \frac{5}{60})\}$, where the membership function has been defined in terms of the frequencies of students who obtained the corresponding grades. Then formula (1) gives that H = $-\frac{1}{\ln 5}(\frac{5}{60}\ln\frac{5}{60}+\frac{12}{60}\ln\frac{12}{60}+\frac{20}{60}\ln\frac{20}{60}+\frac{18}{60}\ln\frac{18}{60}+\frac{5}{60}\ln\frac{5}{60}) \approx$ 0.9 nats

Formula (1) calculates the, so called, *probabilistic uncertainty* of the fuzzy system. The *fuzzy probability* of each element s of U with respect to the corresponding FS is defined by the formula $p_s = \frac{m_s}{\sum_{s\in U} m_s}$.

However, the British economist Shackle (1961) and many other experts after him claimed that the mechanisms of *human reasoning* can be better described by the *possibility* rather than the probability theory. In a fuzzy environment the possibility of each element s of U with respect to the corresponding FS is defined by the formula

$r_s = \frac{m_s}{\max\{m_s\}}$, expressing the "relative membership degree" of s with respect to the maximal membership degree (max $\{m_s\}$) of all the elements of U.

According to Klir and Folger (1988), within the domain of possibility theory the *total uncertainty (TU)* is calculated by the sum of the *strife* or *discord (ST)* expressing conflicts among the various sets of alternatives, and of the *non-specificity* or *imprecision (N)*, which indicates that some alternatives have been left unspecified, i.e. it expresses conflicts among the sizes of the various sets of alternatives. The following examples can serve for obtaining a better understanding of the above two types of *possibilistic uncertainty*:

Example 3: Let U be the set of integers from 0 to 140 representing human ages and let Y = young, A = adult and O = old be FS in U defined by the membership functions m_Y, m_A and m_O respectively. People are considered as being young, adult or old according to their outer appearance. Then, given x in U, there exists frequently a degree of uncertainty about the values that the membership degrees $m_Y(Fox)$, $m_A(Fox)$ and $m_O(Fox)$ could take, resulting to a conflict among the FS Y, A and O of U. For instance, if x=18, values like $m_Y(Fox)$=0.8 and $m_A(Fox)$=0.3 are acceptable, but they are not the only ones. In fact, values like $m_Y(Fox)$=1 and $m_A(Fox)$=0.5 could be also acceptable, etc. This illustrates the type of uncertainty called strife, which is obviously a kindred concept of what it was previously termed as vagueness.

Example 4: In connection to the previous example non-specificity is related to the question: How many x in U should have non zero membership degrees in Y, A and O respectively? Taking into account that the *cardinality* of a FS is defined to be the sum of all membership degrees of the elements of U in it, it becomes evident that the existing in this case uncertainty creates a conflict among the cardinalities, or in other words among the sizes, of the FS Y, A and O. Obviously non-specificity is a kindred concept of ambiguity

Strife and non-specificity are defined on the ordered possibility distribution

r: $r_1=1 \geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$ of the corresponding FS and, taking α =2 for the basis of the logarithms, they are calculated by the formulas (Klir, 1995)

$ST(r)=\sum_{i=2}^{m}(r_i - r_{i+1})\log_2\frac{i}{\sum_{j=1}^{i} r_j}$ (2) and $N(r)=\sum_{i=2}^{m}(r_i - r_{i+1})\log_2 i$ (3) respectively.

Example 5: Calculate the total possibilistic uncertainty of the student class T of Example 2.

Solution: Since max $\{m_s\} = \frac{20}{60}$ the ordered possibility distribution of the FS T is

$r: r_1=1>r_2=\frac{20}{60}>r_3=\frac{18}{60}>r_4=r_5=\frac{5}{60}$.

Therefore, applying formula (2) for n=4 and taking into account that $r_4 = r_5$ one gets that $ST(r)$=

$$(r_2 - r_3)\log_2 \frac{2}{r_1 + r_2} + +(r_3 - r_4)\log_2 \frac{3}{r_1 + r_2 + r_3} = \frac{2}{60}\log_2 \frac{3}{2} + \frac{13}{60}\log_2 \frac{90}{49} \approx 0.299 \text{ bits}$$

In the same way formula (3) gives that $N(r) \approx 0.377$ bits. Therefore $TU(r) \approx 0.676$ bits

Remark: The measurement of a fuzzy system's uncertainty has been used in earlier works of the present author (e.g. see Chapter 5 in Voskoglou (2017) to compare the *mean performance* of two or more different fuzzy systems (e.g. student groups), evaluated by linguistic grades, during the same activity. Nevertheless, the above comparison becomes possible provided that the existing in the systems uncertainty before the activity is the same In this case the lower is a system's uncertainty after the activity, the better its performance. However, frequently the above condition does not hold in real life situations. For example, on comparing the performance of two different student classes in a common exam, one must assume that the average abilities of the students of the two classes are the same, which is not always true.

5. The COG Defuzzification Technique

The solution of a problem using principles and methods of FL involves the following steps:

- *Choice* of the set of the discourse.
- *Fuzzification* of the problem's data by introducing the proper membership functions to define the necessary for this reason FSs.
- *Evaluation* of the fuzzy data by using principles and methods of FL in order to find the solution of the given problem in the form of a *unique* FS.
- *Defuzzification* of the problem's fuzzy solution, i.e. representation of it with a crisp numerical value, in order to be expressed in the natural language.

Among the, at least 30, known defuzzification methods two are the most frequently used in practical applications. In the *Maximum method* one of the values at which the FS expressing the problem's solution has its maximum truth is chosen as the crisp value for the solution's output. For example, this technique is used when the *Bellman-Zadeh's criterion* is applied for solving decision-making problems under fuzzy conditions (e.g. see Chapter 4 in Voskoglou (2017) pp. 97-99).

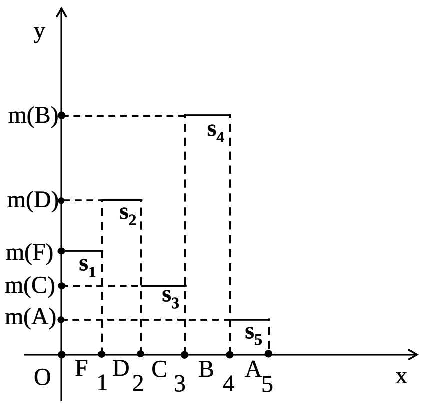
On the other hand, *the Centre of Gravity (COG)* is perhaps the most popular defuzzification technique of fuzzy mathematics. According to it, the crisp value for the solution's output is obtained by calculating the coordinates of the COG of the level's area S contained between the graph of the corresponding membership function and the semi-axis OX (van Broekhoven and Debaets, 2006).

In an earlier work Voskoglou (1999) developed a fuzzy model for a mathematical description of the process of learning a subject matter in the classroom and later on he calculated the existing in the class probabilistic/possibilistic uncertainty to obtain a measure of the student mean performance (e.g. see Chapter 5 in Voskoglou (2017). Meanwhile, Subbotin *et al.* (2004), based on the Voskoglou's model, adapted properly the COG defuzzification technique to be used as an assessment method of student learning skills. Since then, Subbotin and Voskoglou have applied jointly and separately to each other the COG method in many other assessment problems (e.g. see Chapter 6 in Voskoglou (2017). Here a brief account of this method will be presented omitting (but giving references for) some technical details, which are more or less elementary.

Let U be the set of the linguistic grades that have been introduced in Example 2. Then a student group, say G, can be represented as a FS in U in the form

G = {(x, $m(x)$): $x \in U$ }, where y=$m(Fox)$ is the corresponding membership function. Let us assign to each linguistic grade of U a real interval of numerical values as follows: F $\to$ [0, 1), D $\to$ [1,2), C $\to$ [2,3), B $\to$ [3, 4) A $\to$ [4, 5]. This means that $m(Fox)$=m(F) for all x in [0, 1), $m(Fox)$=m(D) for all x in [1, 2), etc. This manipulation enables the design of the graph of the membership function y=$m(Fox)$ (Figure 1), where the area S between the graph and the OX semi-axis is equal to the sum of the areas of the rectangles S_i, i=1, 2, 3, 4, 5, with one side of each one of them lying on OX and having length equal to 1.

Figure-1. The graph of the COG technique



Using the known from Mechanics formulas calculating the coordinates of the COG of plane figures it is straightforward to check that the coordinates of the COG of the area S of Figure 1 are given by the formulas

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$

$$y_c = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right) \quad (4)$$

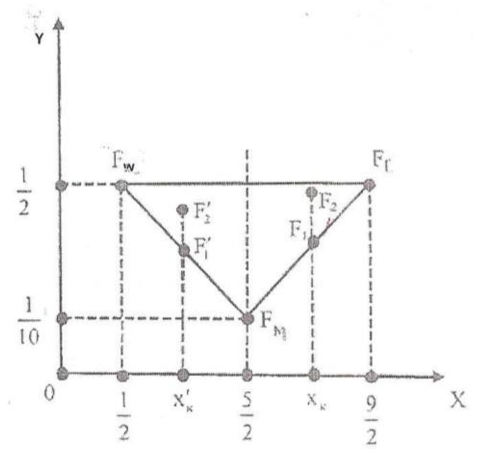
with $y_i = \frac{m(x_i)}{\sum_{x \in U} m(x)}$, where $y_1 = m(F)$, $y_2 = m(D)$,, $y_3 = m(C)$, $y_4 = m(B)$ and $y_5 = m(A)$ (Voskoglou, 2017, Chapter 6, pp. 127-128). Thus $y_1 + y_2 + y_3 + y_4 + y_5 = 1$.

In case of the *ideal* group's performance ($y_5 = 1$ and $y_i = 0$ for i = 1, 2, 3, 4) equations (4) give that the coordinates of the COG, say F_I, are $(\frac{9}{2}, \frac{1}{2})$. In the same way one finds that the COG, say F_W, in case of the *worst* group's performance ($y_1 = 1$ and $y_i = 0$ for i = 2, 3, 4, 5) has coordinates $(\frac{1}{2}, \frac{1}{2})$.

Further, making repeated use of the inequality $a^2+b^2 \geq 2ab$ and adding by members, it is straightforward to check that $5(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) \geq (y_1 + y_2 + y_3 + y_4 + y_5)^2 = 1$, or $y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 \geq \frac{1}{5}$ (Voskoglou, 2017, p. 128).Therefore, the second of equations (4) gives that $y_c \geq \frac{1}{10}$ with the equality holding if, and only if , $y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{5}$.

Consequently, by the first of (4) one gets that the the unique minimum for y_c corresponds to the COG $F_M(\frac{2}{5}, \frac{1}{10})$.

Figure-2. The area in which the COG lies



In concluding the COG of S lies in the area of triangle $F_IF_WF_M$ (Figure 2) and by elementary geometric observations on it one obtains the following assessment criterion :..

- The greater is the value x_c of the x-coordinate of the COG, the better the corresponding group's performance.
- For two groups having the same x_c, if $x_c \geq \frac{5}{2} (< \frac{5}{2})$, then the group with the greater (smaller) value y_c of the y-coordinate of the COG demonstrates the better performance.

Combining equations (4) with the above criterion one concludes that the COG technique is a weighted average of a group's overall performance, since it assigns greater coefficients (weights) to the higher grades. Therefore it measures not the mean, but the *quality performance* of a student group.

Remarks: 1) The above assessment method was called *Rectangular Fuzzy Assessment Model (RFAM)* due to the shape of the graph of the corresponding membership function. Subbotin has also developed variations of RFAM by changing the shape of the graph of the membership function, all of which have finally been proved to be equivalent to the original RFAM (e.g. see Chapter 6 in Voskoglou (2017).

2) A popular method for assessing a group's quality performance under the laws of the traditional logic is the calculation of the *Grade Point Average (GPA}* index. For this, let *n* be the total number of the group's students and let n_x be the number of students who obtained the grade x in U, Then GPA = $\frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n}$ (5) (Voskoglou, 2017).

One can define the membership function of RFAM by $y = m(Fox) = \frac{n_x}{n}$, for all x in U. Then equation (5) can be written as GPA= $y_2 + 2y_3 + 3y_4 + 4y_5$. Therefore the first of equations (4) can be written as $x_c = \frac{1}{2}[(2y_2+4y_3+6y_4+8y_5)+(y_1+y_2+y_3+y_4+y_5)] = \frac{1}{2}(2\text{GPA} + 1)$, or $x_c = \text{GPA}+\frac{1}{2}$ (6).

The last equation combined with the first case of the assessment criterion of RFAM shows that, if the values of the GPA index are different for two groups, then RFAM and GPA provide the same assessment conclusions.

The following example (Subbotin and Voskoglou, 2016) shows that, in case of the same GPA values, the application of the GPA index could not lead to logically based conclusions. Therefore, in such situations the RFAM becomes useful due to its logical nature.

Example 6: Table 2 depicts the student performance of two Classes, say I and II. Apply GPA and RFAM to evaluate their quality performance

Table-2. Student performance

Grades	Class I	Class II
C	10	0
B	0	20
A	50	40

The GPA index for the two Classes is GPA= $\frac{2*10+4*50}{60} = \frac{3*20+4*40}{60} = \frac{22}{6}$. Therefore, according the GPA criterion, the two classes demonstrated the same quality performance.

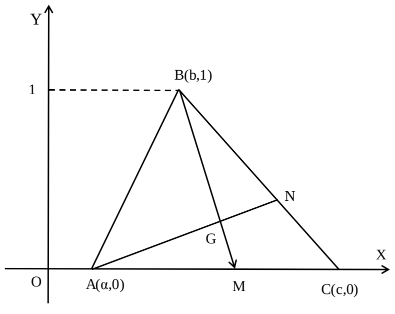
On the other hand, equation (6) gives for RFAM that the x-coordinate of the COG is $x_c = \frac{25}{6} > \frac{5}{2}$ for both Classes. But the second of equations (4) gives that $y_c = \frac{13}{36}$ for Class I and $y_c = \frac{10}{36}$ for Class II. Therefore, according to the second case of the assessment criterion of RFAM, Class I demonstrated a better performance than Class II. Now which one of the above two conclusions is closer to the reality? For answering this question, let us consider the *quality of knowledge*, i.e. the ratio of the students received B or better to the total number of students, which is equal to $\frac{5}{6}$ for the first and 1 for the second Class. Therefore, from the common point of view, the situation in Class II is better. Nevertheless, many educators could prefer the situation in Class I having a greater number of excellent students. However, in no case it is logical to accept that the two Classes demonstrated the same performance, as the calculation of the GPA index suggests.

6. Triangular fuzzy numbers

6.1. Background

Fuzzy Numbers (FNs), which are a special form of FSs in the *set* ***R*** of real numbers, play an important role in fuzzy mathematics, analogous to the role of the ordinary numbers in crisp mathematics. The basic arithmetic operations of real numbers can be extended to FNs with two, equivalent to each other, methods. For the definition, the several forms, the properties and operations and other general facts on FNs we refer to the book of Kaufmann and Gupta (1991). Here, we shall focus on the *Triangular FNs (TFNs)*, being the simplest form of FNs,

Figure-3. Graph and COG of the TFN (a, b, c)



A TFN (a, b, c), with a, b, c in ***R***, $a<b<c$, represents mathematically the fuzzy expression "b lies in the interval [a, c]". Its membership function's graph forms a triangle with the x-axis in the interval [a, c], whereas it is annihilated outside [a, c] (Figure 3).Therefore, the analytic definition of (a, b, c) is the following:

Definition: Given the real numbers a, b and c, with $a < b < c$, the TFN (a, b, c) is a FS in ***R*** with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \ , \ x > c \end{cases}$$

Consequently $m(b) = 1$.

If $A = (a, b, c)$, $B = (a_1, b_1, c_1)$ are given TFNs and k is a positive real number, then, applying the general methods of defining arithmetic operations among FNs, it can be shown that:

- The *sum* $A + B$ is the TFN $(a+a_1, b+b_1, c+c_1)$.
- The *difference* $A – B$ is the TFN $(a-c_1, b-b_1, c-a_1)$.
- The scalar product kA is the TFN (ka, kb, kc).
- The *product* $A.B$ and the *quotient* $A : B$ are FNs, which are not always TFNs.

The following result can be used for the defuzzification of TFNs with the COG technique:

Proposition: The coordinates (X, Y) of the COG of the graph of the TFN (a, b, c) are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$ (7).

Proof: The graph of (a, b, c) is the triangle ABC of Figure 3, with A (a, 0), B (b, 1) and C (c, 0). The COG, say G, of ABC is the intersection point of its medians AN and BM, where N($\frac{b+c}{2}, \frac{1}{2}$), M($\frac{a+c}{2}$, 0). Therefore the coordinates of G can be easily calculated by routine methods of the Analytic Geometry.

When the student individual performance is assessed with qualitative instead of numerical grades, the traditional method of calculating the average of the student scores cannot be applied for evaluating the *mean performance* of a student group. A method using TFNs has been developed for such cases in earlier works of the present author (e.g. see Voskoglou (2017), Chapter 7). For this, the following definition has been introduced:

Let $A_1, A_2, \ldots, A_n$, be a finite number of given TFNs. Then, their *mean value* is defined to be the TFN $A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n)$.

The following example illustrates the above method:

Example 7: Table 3 depicts the individual student performance of two groups, T_1 and T_2. Estimate the mean performance of the two groups.

Table-3. Student performance

Grade	T_1	T_2
A	20	20
B	15	30
C	7	15
D	10	15
F	8	5
Total	60	85

Solution: Introducing the numerical scale of scores from 1 to 100 we assign to each of the grades A, B, C, D, F a TFN, denoted by the same letter in italic fond, as follows:

A (85, 92.5, 100), *B* (75, 79.5, 84), *C* (60, 67, 74), *D* (50, 54.5, 59) and *F* (0, 24.5, 49). Observe that the middle term of each of those TFNs is equal to the mean value of its other two terms. Therefore, according to Table 3, we have 60 and 85 in total TFNs for the groups T_1 and T_2 respectively representing their students' individual performance. Calculating the mean values of those TFNs, denoted by the same letters in italic fond, one finds that $T_1 = \frac{1}{60}(20A+15B+7C+10D+8F) \approx (62.42, 70.88, 79.33)$ and

$T_2 = \frac{1}{80}(20A+30B+15C+15D+5F) \approx (65.88, 72.71, 79.53)$ respectively.

It is logical to accept that the mean performance of the two groups can be estimated those TFNs. Defuzzifying them one finds that the x-coordinates of the COGs of their graphs are $X(T_1) = \frac{62.42+70.88+79.33}{3} \approx 70.88$ and similarly $X(T_2) \approx 72.71$. Therefore both groups demonstrate a good (C) mean performance with the performance of T_2 being better.

Remark: Another kind of FNs that we have used in assessment problems (e.g. see Chapter 7 in Voskoglou (2017) are the *Trapezoidal Fuzzy Numbers (TpFNs)*, which are a generalization of TFNs. Recently (Voskoglou and Theodorou, 2017) we have also developed an assessment method using *Grey Numbers (GNs)***.** Although this method is equivalent to that using the TFNs, it reduces significantly the computational burden. Note that the GNs and their arithmetic are defined with the help of the real intervals.

7. Conclusion

The effectiveness of a method to deal with situations of uncertainty is characterized by its potential to solve problems connected to the various forms of the uncertainty. Under this skeptic, FL exceeds probability, since its application enables the solution of problems which can not be solved with the traditional methods of probability theory. Nevertheless, fuzzy models are greatly based to the subjective perception of their designers (e.g. definition of the membership functions, choice of the suitable defuzzification method, etc.). Therefore the creditability of fuzzy models in representing the corresponding real situations must be checked with a great attention.

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