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The Effects of Formalism and Intuitionism on the Development of Mathematics Education

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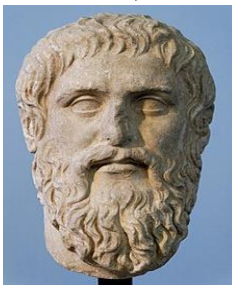
Abstract

Two extreme philosophies about its orientation have been tacitly appeared almost from ithe origin of mathematics as an autonomous science: Formalism**,** where emphasis is given to the content and intuitionism, where the attention is turned to problem-solving processes. In this article the effects of the above two philosophies on the development of mathematics education are studied. Crucial problems for the future of mathematics education are also discussed, such as the role that computers could play for the teaching and learning of mathematics, etc. Although neither formalism nor intuitionism have finally succeeded to find a solid framework for mathematics, most of its recent advances were obtained through their disputations about the absolute mathematical truth. On the contrary, these disputations have created serious problems in the sensitive area of mathematics education, the most characteristic being probably the failure of the introduction of the "New Mathematics" to school curricula that distressed students and teachers for many years. Therefore, there is a need for those working in the area of mathematics education to search for a proper balance among the several philosophical aspects of mathematics. This will bring the required tranquillity in the area, in order to be developed smoothly for the benefit of future generations.
Keywords: Philosophy of mathematics; Platonism; Paradoxes of Set theory; Formalism; Intuitionism; Mathematics education; Problem-solving; Mathematical modeling; Computers in the teaching and learning of mathematics.

1. Introduction

The scientific beliefs about the nature .of mathematics were focused for centuries on the Plato's (Picture 1) view about the existence of an abstract, eternal and unchanged universe of mathematical forms. Consequently it was strongly believed that mathematics is not invented, but it is gradually discovered by humans (*Platonism*). In a more general context, all those who believe that mathematics exists independently from the human mind belong to the school of *mathematical realism* and they are divided into several categories with respect to their beliefs about the texture of the mathematical entities and the way in which we learn it (Voskoglou M. Gr., 2017).

Picture-1. Plato (424-377 BC)



However, the radical advances on Mathematics during the last two centuries, such as the appearance of the non Euclidean Geometries, the axiomatic foundation of the Set Theory that enables one to consider four different forms of it (see Section 2), the proof of the Gödel's Incompleteness Theorems, the eventual enrolment of informatics in the pure mathematical research, etc., as well as data collected from experimental researches of cognitive scientists and psychologists, have currently turned to a great deal the scientific views to the belief that mathematics is actually an *invention* of the human mind (Voskoglou M. Gr., 2017). The *embodied mind theories,* for instance, hold that mathematical thought is a natural outgrowth of the human cognitive apparatus, which finds itself in our physical universe; e.g. the abstract concept of number springs from the experience of counting discrete objects. Thus humans construct and do not discover, mathematics. There also exist intermediate theories stating that mathematics is a *mixture* of human *inventions (axioms, definitions)* and of *discoveries (theorems)*, (Livio, 2006).

In such a dynamic environment of contravening ideas about the nature of mathematics the *philosophy of mathematics* was rapidly developed and the known *schools of mathematical thought* were gradually established with their typical forms. It is recalled that the philosophy of mathematics is the branch of philosophy that studies the assumptions, foundations, and implications of mathematics, and aims to provide a viewpoint of the nature and methodology of mathematics, and to understand the place of mathematics in people's lives (Wikipedia, 2018).

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The target of the present work is to investigate the influence (positive and negative) of formalism and intuitionism, the two main schools of mathematical thought, on the development of mathematics education. The rest of the article is formulated as follows: The main ideas of formalism and intuitionism are briefly exposed in Section 2, whereas Section 3 deals with the main focus of the article by examining how the ideas of these two schools have affected the sensitive area of mathematics education. Crucial problems for the future of mathematics education are discussed in Section 4, such as the role that computers could play for the teaching and learning of mathematics, etc. Finally, the general conclusions of this study are stated in Section 5.

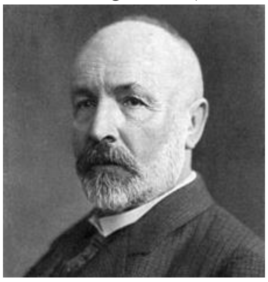
2. Formalism and Intuitionism in Mathematics

Two extreme philosophies about the orientation of mathematics have been tacitly appeared almost from its origin as an autonomous science: F*ormalism,* where emphasis is given to the content and i*ntuitionism,* where the attention is turned to problem-solving processes.

The axiomatic foundation of Geometry in Euclid's *"Elements"*, the most famous in the world mathematical classic, is a characteristic example of the formalistic point of view. An analogous example for intuitionism is the less known to the West world Oriental counterpart *"Jiu Zhang Suan Shu" (Nine Chapters on Mathematics)* (Ma, 2005). Although very different in form and structure from Euclid's "Elements', it has served as the foundation of traditional Oriental mathematics and it has been used as a mathematics text book for centuries in China and in most other countries of Eastern Asia. Its title has been translated to English in various ways. Although "mathematics" seems to be a more accurate translation of "Suan Shu" than mathematical art, it seems that mathematics in the East was indeed more of an art as compared to mathematics in the West as a science. Very many centuries later, during the 19th and the beginning of the 20th century, the *paradoxes* appeared in *Set Theory* was the reason of an intense dispute among the followers of the two philosophies, which however was extended much deeper into the mathematical thought.

The set theory has been proved to be fundamental for the development of the whole specter of mathematics resulting to the foundation of its several branches on a more solid basis and to their enrichment with new ideas and directions. It is recalled that the founder of the set theory Cantor (Picture 2) defined the concept of a set as a finite or infinite collection of objects (elements) of any nature, different to each other, sharing a common characteristic property, so that they can be considered as a totality. It becomes therefore evident that a set cannot simultaneously be one of its elements. However, our unlimited capability of creating any kind of new sets can easily lead to paradoxes. For example, the *set of all the sets* is obviously an element of its self! Also, if T is the set of all sets that they do not contain themselves as an element, then obviously $T \in T$ implies that $T \notin T$ and $T \notin T$ implies that $T \in T$ (*Russel's paradox*)! Of course the catalogue of the paradoxes is not completed here (Breuer, 2006) and such an attempt is out of the scope of this article.

Picture-2. Georg Cantor (1845-1918)



The important thing for our focus is that the paradoxes of the set theory gave the impulsion to the German mathematician Ernst Zermelo (1871-1953), following the road opened by Euclid for Geometry many centuries ago, to introduce in 1908 a way of restating the Set Theory in terms of a system of axioms. As a result, the paradoxes were by-passed through a careful statement of those axioms so that to blockade contradictory notions like the set of all the sets, etc. The axiomatic system of Zermelo was enriched by Livio (2006), and was further improved by Livio (2006), so that everything seemed to work well. But gradually, one of the axioms started to cause headache to the mathematicians. This was *the axiom of choice,* according to which, if X is a set of non empty sets, then one can choose a unique element from each of these sets in order to create a new set Y. When X is either a finite set, or it is an infinite set but we know the rule under which the choice is made, then the above statement works well. The problem is located when X is an infinite set and the rule of the choice is unknown. In this case the choice does never end and the existence of Y becomes a matter of faith rather than a reality. For example, assuming that X is an infinite set of pairs of shoes, if we decide to choose always the right shoe from each pair, then there is no problem. On the contrary, if X is an infinite set of pairs of stockings, then obviously we have problem with the choice.

This problem made the mathematicians to start thinking, as it had happened centuries ago with the fifth Euclid's axiom, if the axiom of choice could be either proved or by-passed with the help of the other axioms. The answer to this question was partially given by Gödel (Picture 3), who proved that the axiom of choice as well as the Cantor's *continuum hypothesis* [1] are consistent to the rest of the Zermelo - Fraenkel axioms; i.e. they cannot be contradicted

[1] The continuum hypothesis, which was the first of the 23 unsolved mathematical problems presented in 1900 by Hilbert at the International Conference of Mathematics in Paris, states that the set of real numbers has the minimal cardinality which is greater than the cardinality of the set of non negative integers. Moreover, *the generalized continuum hypothesis* states that the cardinality of the power set of each infinite set is the smaller cardinality which is greater than the cardinality of this set.

by them (Gödel, 1940). Moreover, for the continuum hypothesis this remains true even if the axiom of choice is added to the other Zermelo - Fraenkel axioms. The Gödel's result was completed by the American mathematician Paul Cohen, who proved in 1963 that the axiom of choice and the continuum hypothesis cannot be proved by the other axioms of set theory and that this is true for the continuum hypothesis even if the axiom of choice is added to those axioms. The combination of the Cohen's and the Gödel's results show that the axiom of choice and the continuum hypothesis are independent from the other axioms of set theory. Therefore, considering the continuum hypothesis as an axiom and adding it to the system of the Zermelo - Fraenkel axioms, one can create four different theories for the sets: The first one by including to it both the axioms of the choice and of the continuum, the next two by including only one of them in each case and the fourth one by including none of them! Therefore, the open "war" between formalism and intuitionism had already been started without any mercy!

Picture-3. Curt Gödel (1906 – 1978)

Formalism on the one hand claims that the mathematical statements may be thought of as statements about the consequences of certain string manipulation rules. For example, Euclidean geometry is seen as consisting of some strings called "axioms", and some "rules of inference" to generate new strings (theorems) from the given ones. Apart from the *axiomatic foundation of mathematics*, the main beliefs of formalism include the need of *consistency* of the axioms and the notions not permitting the creation of absurd situations, the Aristotle's *law of the excluded middle* (something is either true or false) and the *possibility of the existence of a solution* (positive or negative) for each mathematical problem, even if such a solution has not been found yet. For example, let A be the set of all sets. Then A = A and also A $\neq$ A, since A belongs to A and therefore A is a proper subset of A. But this is absurd, which means that the notion of the set A of all the sets is not consistent and therefore it does not exist

The main critique against formalism is that the genuine ideas and inspirations that occupy mathematicians are far removed from the manipulation games with the stings of axioms mentioned above. Formalism is thus silent on the question of which axiom systems ought to be studied, as none of them is more meaningful than another from the formalistic point of view.

The program of the leader of formalism David Hilbert (Picture 4) aimed to a complete and consistent axiomatic development of all branches of mathematic. However, the *Gödel's incompleteness theorems* put a definite end to his ambitious plans. In fact, there is no system that can prove the consistency of another system, since it has to prove first its own consistency, which, according to the second of the above theorems it is impossible! Therefore, the best to hope is that the statement of a certain system's axioms, although by the first Gödel's theorem it cannot be complete, it is consistent.

Picture-4. D. Hilbert (1862-1943)

On the other end, the main beliefs of *intuitionism* include the *primitive understanding of the natural numbers* (for the formalists proofs are needed for the consistency of the arithmetic operations among them) and the connection of the mathematical existence of an entity with the possibility of *constructing* it. For example, Zermelo proved that each non empty set can be well ordered, i.e. can be ordered in such a way that each subset of it has a minimal element. However, for the intuitionists this theorem has not any value, since it does not suggest the way in which such an order could be constructed.

In addition, intuitionists do not accept neither the law of the excluded middle, nor the possibility of the existence of a solution for each problem, although problems without a positive or negative solution have not been appeared in the history of mathematics until now. Further, the formalistic view that a notion's consistency guarantees its existence is completely unacceptable for the intuitionists. L. Kronecker (1823- 1891), the main pioneers of intuitionism, used to say that "God created the natural numbers, whereas all the other mathematical entities have been created by the humans". The leader of intuitionism was L. E. J. Brouwer (Picture 5), whereas H. Weyl (1885-1965) and several others have played also an important role.

Picture-5. L. E. J. Brouwer (1881-1965)

In intuitionism, the term "construction" is not cleanly defined, and that has led to criticisms. Attempts have been made to use the concepts of Turing machine or computable function to fill this gap, leading to the claim that only questions regarding the behaviour of finite algorithms are meaningful and should be investigated in mathematics. This has led to the study of the *computable numbers*, first introduced by Turing (1936) that are associated with the theoretical computer science.

The study of the history of mathematics reveals that there exists a continuous oscillation between formalism and intuitionism (Davis and Hersh, 1981). This oscillation is symbolically sketched in Figure 1, where the two straight lines represent the two philosophies, while the continuous broadening of space between the lines corresponds to the continuous increase of mathematical knowledge. According to Verstappen (1988) the period of this oscillation is of about 50 years, which has been also crossed by Galbraith (1988) by studying a diagram, due to Shirley, representing a parallel process between the alterations of the economical conditions and the changes appearing to the mathematical education systems of the developed west countries

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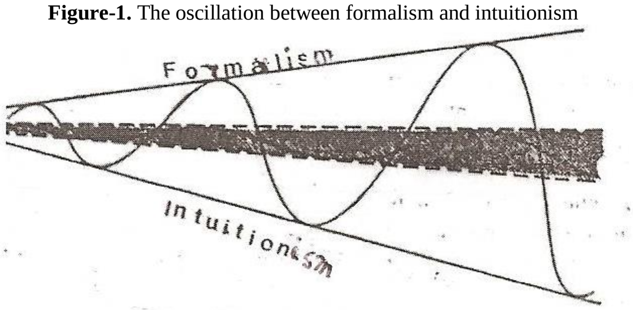
Figure-1. The oscillation between formalism and intuitionism



Examples of how the "mathematics pendulum" swung from the one extreme to the other over the span of about a century, include the evolution from the purely axiomatic mathematics of the *school of Bourbaki* to the reawaking of experimental mathematics, from the complete banishment of the "eye" in the theoretical hard sciences to the computer graphics as an integral part of the process of thinking, research and discovery and also the paradoxical evolution from the invention of "pathological monsters", such as Peano's curve or Cantor's set – which Poincare said that should be cast away to a mathematical zoo never to be visited again – to the birth of Mandelbrot (1983) *Fractal Geometry of Nature*. To Mandelbrot's surprise and to everyone else's, it turns out that these strange objects, coined fractals**,** are not mathematical anomalies but rather the very patterns of nature's chaos!

Apart from the above two extreme philosophies and the ideas of mathematical realism and of the embodied mind theories mentioned in our Introduction, several other schools of mathematical thought have been appeared in the history of mathematics, each one having its own strengths and weaknesses. *Logicism,* for example, developed in the beginning of the 20th century, believes that mathematics is reducible to logic, and hence it is nothing but a part of logic (Shapiro, 2000). Also *structuralism* is a more recent position holding that mathematical theories describe structures, and that mathematical objects are exhaustively defined by their places in such structures; e.g. the real numbers are completely defined by their places in the real line (Shapiro, 2000). The catalogue of the schools of mathematical thought does not end here (Shapiro, 2000; Wikipedia, 2018), but a complete reference to all of them is out of the scope of the present work, which is focusing on the effects of formalism and intuitionism on mathematics education.

3. The Influence of Formalism and Intuitionism on Mathematics Education

The traditional components of school mathematics, i.e. Arithmetic, Euclidean Geometry, Trigonometry and Elementary Algebra, had remained stable for many years, almost from the time of Napoleon the Great! However, as a consequence of the "mathematics pendulum" swing, dramatic changes also happened in the area of mathematics education during the last 50-60 years. First, the result of the post – war effort to bring mathematics as a teaching subject into harmony with mathematics as a science, as it has been developed since the last quarter of the 19th century, with an increasing gap between school mathematics and modern higher level mathematics, was the introduction, during the 60's, of the *"New Mathematics"* in the curricula of studies. New chapters were added in the curricula, like Set Theory, elements from Linear and Abstract Algebra (matrices, determinants, algebraic structures, etc.) and Mathematical Logic, Probability and Statistics and of course Mathematical Analysis up to the study of integrals in one variable and even of simple forms of Differential Equations.

The way of presentation of the material was also changed, since the traditional inductive methods involving many examples and applications gave their place to a strict, axiomatic presentation that created many difficulties not only to the students, but also to the teachers, who were not adequately prepared to teach the new topics introduced in the curricula. Moreover, the volume of the total amount of the material to be taught was enormously increased, since some space should be also found for the old, traditional school mathematics.

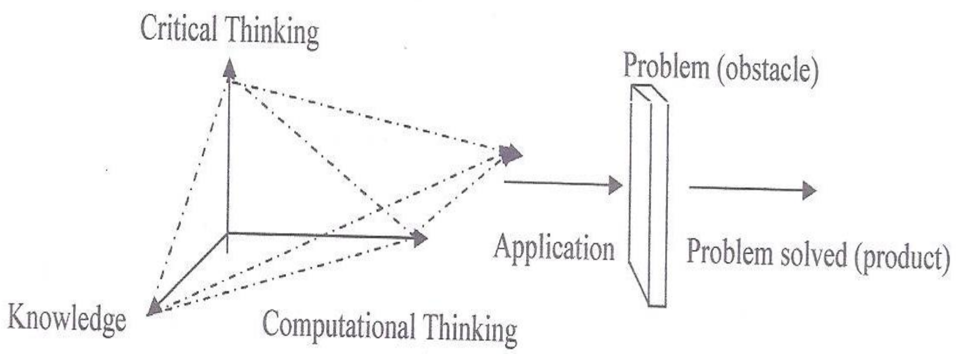
Therefore, it did not take many years to realize that the new curricula did not function satisfactorily all the way through, from primary school to university, even if the problems varied with the level (Kline, 1973). Thus, and after the rather vague "wave" of the *"back to the basics"*, considerable emphasis has been placed during the 80's on the use of the problem as a tool and motive to teach and understand better mathematics (Voskoglou M. Gr., 2011), with two main components: *Problem – Solving,* where emphasis was given to the use of *heuristics* (solving strategies) for the solution of mathematical problems (Polya, 1945; Schoenfeld, 1980) and *Mathematical Modelling and Applications*, dealing with the formulation and solution of a special type of mathematical problems generated by corresponding problems of the real world and of the everyday life (Pollak, 1979; Voskoglou M. Gr., 2015a). The attention was turned also to *Problem - Posing,* i.e. to the process of extending existing or creating new problems (Brown and Walters, 1990).

The excessive emphasis given during the 80's on the use of the heuristics for problem-solving received several critiques (Lawson, 1990; Owen and Sweller, 1989) suggesting that the attention should be turned rather to the presentation of well prepared examples (solved problems) ant to the automation of rules. The argument was that these approaches facilitate better the *transfer of knowledge* and the acquisition by the students of the proper *schemas***,** than the analytic methods of the problem-solving strategies that impose a heavy cognitive weight on students. On the contrary, mathematical modelling has been evolved nowadays to a teaching method of mathematics, usually referred as *application-oriented teaching of mathematics* (Voskoglou M. G., 2005).

A current approach of mathematics education is the utilization of *informatics* as a tool for the teaching and learning of mathematics. In fact, the animation of figures and of mathematical representations obtained by using suitable mathematical software, increases the students' imagination and helps them to find easier the solutions of the corresponding problems. The role of mathematical theory after this is not to convince, but to explain.

Moreover, by thinking like a computer scientist, students become aware of behaviours and reactions that can be captured in algorithms or can be analysed within an algorithmic framework. *Computational thinking*, the modern expression of algorithmic thinking (Wing, 2006), gives them nowadays a different framework for visualizing and analyzing, a whole new perspective of solving strategies. Figure 2 (Voskoglou M. Gr. and Buckley, 2012), represents how the two basic modes of thinking, i.e. computational and *critical thinking***,** are combined with the existing mathematical knowledge to solve a complicated problem. This representation is based on the approach that, when the already existing knowledge is adequate, the necessary for the problem's solution new knowledge is obtained through critical thinking, while computational thinking is applied to design and to execute the problem's solution.

Figure-2. Computational thinking in Problem-Solving



In concluding, critical thinking is a prerequisite to knowledge acquisition and to its application to solve problems, but not a sufficient condition when one face complex problems of the real life (e.g. technological problems), which require also a pragmatic way of thinking for their solution, such as the computational thinking is.

4. Discussion

In the name of the introduction of modern mathematical topics in the school curricula, like Mathematical Analysis, Analytic Geometry, Probability Theory, etc., the teaching of the traditional *Euclidean Geometry* has drastically restricted and neglected. For example, nowadays we have reached to the point that in the Greek Upper High School (Lyceum) the 3-dimensional Euclidean Geometry is usually not taught at all! This approach, according to the opinion of the majority of educators and researchers in the area of mathematics education, including the present author, is a big pedagogical mistake. In fact, although the traditional Geometry is nowadays out of the focus of the modern mathematical research, it remains an indispensable pedagogical tool for enhancing the student mathematical thinking and fantasy, since its objects are real, solid and within the student cognitive experiences. That is why many tertiary teachers of mathematics, taking into account the weaknesses of their students in understanding the properties of space, they suggest that it would be much better for them to be taught the geometry of space in the Upper High School, instead of learning the abstract properties of the integrals and other details of Mathematical Analysis, that could be taught more analytically at the university level.

Another problem may be created by the mistaken view of a number of experts and educators that *mathematical modeling* could become a general, i.e. applicable in all cases, method for teaching mathematics. In fact, mathematical modeling has many advantages, because it connects mathematics to real world situations, thus revealing its usefulness to students and therefore increasing their interest for it. However, the attempt to teach everything through mathematical modeling hides the danger to neglect the mathematical content in favor of the applications.

A few years ago, I presented in the ICTMA Newsletter (Voskoglou M. Gr., 2014) two mathematical modeling problems on the use of the derivative for calculating the extreme values of a function in one variable. The one of them was about the construction of a channel to run the maximum possible quantity of water through it, by folding the two edges of a metallic leaf so that to remain perpendicular to the surface of the rest of the leaf. An anonymous critique was published together with it, suggesting that it could be much more interesting, if I had left the choice of the angle of the edges of the leaf to my students. My answer (Voskoglou M. Gr., 2015b) was that, if I had done so, it could be a good exercise on problem-posing, but my students, being busy by playing with the construction of the channel, would probably not learn anything more about the derivatives!

A third and last comment that is of worth to be added here is about the use of the *computers* as a tool in the process of teaching and learning mathematics. Students today, using the convenient small calculators, can make quickly and accurately all kinds of numerical operations. Further, the existence of a variety of suitable mathematical software gives them the possibility to find automatically the solutions of all the standard forms of equations and of systems of equations, to make any kind of algebraic operations, to calculate limits, derivatives, integrals, etc, and even more to obtain all the alternative proofs of a known mathematical theorems without any spiritual effort.. Therefore, a number of experts in computer science have already concluded that in the near future teachers will not be necessary for the process of learning mathematics, because everything will be done by the computers. “The use of the horses became not necessary” they use to parallelize, “from the time that cars have been invented”!

However, this is actually an illusion. In fact, the acquisition of information is important for the learner, but the most important thing is to learn how to think logically and creatively. The latter is impossible, at least for the moment, to be achieved by the computers alone, since computers have been created by the humans and they come into ‘life’ through programming, which was also done by a human being. Thus the old credo “garbage in, garbage out” is still valid. Therefore, although the computers dramatically exceed in speed, most probably they will never reach the quality of the human mind[2]. On the other hand, the practice of students with numerical, algebraic and analytic calculations, through the solution of problems and the rediscovery of the proofs of the existing theorems, it is necessary to be continued for ever; otherwise students will gradually loose the sense of numbers and symbols, the sense of space and time, thus becoming unable to create new knowledge and technology.

Of course, there is no doubt that computation has became nowadays an increasingly essential tool for doing scientific research. The *Artificial Intelligence’s* technologies aim at duplicating the capacity of the human mind by adding the advantage of operating at higher speeds than the mind in computations. It is expected that future generations of scientists and engineers will need to engage and understand computing in order to work effectively with management systems, technologies and methodologies. However, all those are related to the need of finding ways of teaching effectively the informatics and especially the *computer programming* to the future generations of students and not to the teaching of mathematics. In the last area, computers can certainly play the role of a valuable tool that makes the learning process easier and more effective, but in no case they can replace the teachers of mathematics!

5. Conclusion

In the present work we studied the effects on the development of mathematics education of the two main schools of mathematical thought, the formalism and the intuitionism. Crucial problems for the future of mathematics education were also discussed, like the role of the computers in the teaching and learning of mathematics, etc.

Although formalization and intuitionism have not succeeded in finding a solid framework for mathematics, most of the recent advances of this science were obtained through their disputations about the absolute mathematical truth. On the contrary, these disputations have created serious problems in the sensitive area of mathematics education, the most characteristic being probably the failure of the introduction of the “New Mathematics” to school education that distressed students and teachers for many years.

In Chinese philosophy *Yin* and *Yang* represent all the opposite principles (Ma, 2005). It is important however to pay attention to the fact that these two aspects rather complement and supplement than opposing each other, with the one containing some part of the other. This kind of philosophy seems to be a suitable one for the field of mathematical education. In fact, although it is logical for each one of those working in the area to be closer to the ideas of a certain school of mathematical thought, what it is actually needed is to find a proper balance among the ideas of all those schools by accepting their advantages and by pointing out their weaknesses.. In this way the area of

[2] It is true that nowadays a new generation of computers has been created that are programmed to build new computers being better than themselves! However, this does not guarantee at all that eventually they will approach the quality of the human mind.

mathematics education will find the required tranquillity to be developed smoothly for the benefit of the future generations.

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