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Prime Labeling of Newly Constructed Graph Using Star Graphs and Complete Bipartite Graphs

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Abstract

A graph $G = (V(G), E(G))$with $|V(G)|$ vertices is said to have prime labeling if there exist a bijection map $f : V(G) \rightarrow \{1,2,3, \ldots, |V(G)|\}$ such that for each edge $e = uv$ in $E(G)$, *f(u)* and *f(v)* are relatively prime. A graph G which admits prime labeling is called a prime graph. Two integers are said to be relatively prime, if their greatest common divisor $(g.c.d)$ is 1. A complete bipartite graph is a simple bipartite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite set. A $K_{p,q}$ graph is a complete bipartite graph which has p vertices in one partite set and q vertices in other partite set, where $p, q \geq 1$. If $p = 1$, then $K_{1,q}$ graph is called a star graph. The present work focuses on prime labeling on simple finite undirected graphs related to star graph. In our work, we proved the graphs obtained by replacing every edge of a star graph $K_{1,n}$ by $K_{2,m}$ is a prime graph, where $n \geq 1$ for $m = 4, 8, 9$ and $1 \leq n \leq 16$ for $m = 6$.

Keywords: Prime labeling; Prime graph; Complete bipartite graph; Star graph.

1. Introduction

The concept of the graph grew from a problem Leonhard Euler had developed concerning the city of Königsberg. The city of Königsberg, at the time when Euler contemplated, it was located in Prussia. A graph G is a finite nonempty set V of objects, called vertices together with a possibly empty set E of 2-element subsets of V called edges. Graphs are commonly viewed as drawings where the vertices are points or circles and the edges as lines occurring between two vertices. Two distinct vertices u and v are adjacent if the edge $\{u, v\}$ is in contained in the edge set E of G. Rather than denoting the edges as a two element subset of the vertex set, we will shorten the notation to the edge uv for any two connected vertices u and v.

The notation of a prime labeling was introduced by Roger Entringer. Around 1980 Roger Entringer conjectured that all trees having prime labeling which is not settled till today. Fu and Huang proved trees with 15 or fewer vertices are prime in 1994.

A graph $G(V(G), E(G))$ with $|V|$ vertices is said to have prime labeling if its vertices can be labeled with distinct positive integers not exceeding $|V|$ such that the label of each pair of adjacent vertices are relatively prime. Two integers are said to be relatively prime, if their greatest common divisor $(g.c.d)$ is 1. A graph G which admits prime labeling is called a prime graph.

2. Material and Methods

In our research, we consider the graphs obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,m}$ is a prime graph, where $n \geq 1$ for $m = 4, 8, 9$ and $1 \leq n \leq 16$ for $m = 6$.

2.1. Theorem 1

The graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,4}$ is a prime graph, where $n \geq 1$.

Proof:

Let H be the graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,4}$ is a prime graph. Let the vertices of $K_{1,n}$ be $u_0, u_1, u_2, \ldots, u_n$ with u_0 as center vertex and it is label as 1.

Furthermore, every edge $u_0 u_i$ of $K_{1,n}$ replaced by u_{i1}, u_{i2}, u_{i3} and u_{i4} by joining $u_0 u_{i1}, u_{i1} u_i, u_0 u_{i2}, u_{i2} u_i, u_0 u_{i3}, u_{i3} u_i, u_0 u_{i4}$ and $u_{i4} u_i$, for $1 \leq i \leq n$.

Then the new vertex set is $V(G) = \{u_0, u_i, u_{i1}, u_{i2}, u_{i3}, u_{i4}\}$ for $1 \leq i \leq n$ and new edge set is $E(G) = \{u_0 u_{i1}, u_{i1} u_i, u_0 u_{i2}, u_{i2} u_i, u_0 u_{i3}, u_{i3} u_i, u_0 u_{i4}, u_{i4} u_i\}$ for $1 \leq i \leq n$. So $|V(G)| = 5n + 1$, where $n \geq 1$.

Define a labeling function $f: V(H) \rightarrow \{1,2,3, \ldots, 5n + 1\}$ as follows,

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$$f(u_i)=\begin{cases}5i, & i\equiv 1\ mod(6)\\ 5i-1, & i\equiv 0,2\ mod(6)\\ 5i-2, & i\equiv 3,5\ mod(6)\\ 5i-3, & i\equiv 4\ mod(6)\end{cases}$$
$$f(u_{i1})=\begin{cases}5i-3, & i\not\equiv 4\ mod(6)\\ 5i-2, & i\equiv 4\ mod(6)\end{cases}$$
$$f(u_{i2})=\begin{cases}5i-2, & i\not\equiv 3,4,5\ \ mod(6)\\ 5i-1, & i\equiv 3,4,5\ mod(6)\end{cases}$$
$$f(u_{i3})=\begin{cases}5i-1, & i\equiv 1\ mod(6)\\ 5i, & i\not\equiv 1\ mod(6)\end{cases}$$
$$f(u_{i4})=5i+1,\quad 1\le\ i\le n$$

Now, consider Greatest Common Devisor of $f(u_i)$ and other four vertices which are $f(u_{i1}), f(u_{i2}), f(u_{i3})$ and $f(u_{i4})$.

Case 1.
Assume $i\equiv 1\ mod(6)$, so that $f(u_i)=\ 5i$, note that,
$g.c.d(f(u_i), f(u_{i1}))=\ g.c.d(5i, 5i-3)=\ 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i2}))=\ g.c.d(5i, 5i-2)=\ 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i3}))=\ g.c.d(5i, 5i-1)=\ 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i4}))=\ g.c.d(5i, 5i+1)=\ 1$(consecutive positive numbers)

Case 2.
Assume $i\equiv 0,2\ mod(6)$, so that $f(u_i)=\ 5i-1$, note that,
$g.c.d(f(u_i), f(u_{i1}))=g.c.d(5i-1, 5i-3)=1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i2}))=g.c.d(5i-1, 5i-2)=\ 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i3}))=g.c.d(5i-1, 5i)=\ 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i4}))=g.c.d(5i-1, 5i+1)=\ 1$(consecutive odd numbers)

Case 3.
Assume $i\equiv 3,5\ mod(6)$, so that $f(u_i)=\ 5i-2$, note that,
$g.c.d\big(f(u_i), f(u_{i1})\big)=g.c.d(5i-2, 5i-3)=1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i2}))=g.c.d(5i-2, 5i-1)=\ 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i3}))=g.c.d(5i-2, 5i)=\ 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i4}))=g.c.d(5i-2, 5i+1)=\ 1$(are not multiple of 3 and differ by 3)

Case 4.
Assume $i\equiv 4\ \ mod(6)$, so that $f(u_i)=\ 5i-3$, note that,
$g.c.d\big(f(u_i), f(u_{i1})\big)=g.c.d(5i-3, 5i-2)=1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i2}))=g.c.d(5i-3, 5i-1)=\ 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i3}))=g.c.d(5i-3, 5i)=\ 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i4}))=g.c.d(5i-3, 5i+1)=\ 1$(odd integers that differ by 4)

Clearly, vertex labels are distinct. Thus labeling defined above gives a prime labeling for H. Therefore H is a prime graph.

For example, the prime labeling of the graph obtained by replacing every edge of a star graph $K_{1,3}$ by $K_{2,4}$ using labeling appears in figure 1.

2.2. Theorem 2

The graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,6}$ is a prime graph, where $1\le n\le 16$.

Proof:

Let H be the graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,6}$ is a prime graph. Let the vertices of $K_{1,n}$ be $u_0, u_1, u_2, \dots, u_n$ with u_0 as center vertex and it is label as 1.

Furthermore, every edge u_0u_i of $K_{1,n}$ replaced by $u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}$ and u_{i6} by joining $u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}$ and $u_{i6}u_i$, for $1\le\ i\le n$, where $1\le n\le 16$.

Then the new vertex set is $V(G)=\{\ u_0,\ u_i,\ u_{i1},\ u_{i2}, u_{i3},\ u_{i4},\ u_{i5},\ u_{i6}\}$ for $1\le\ i\le n$ and new edge set is,
$E(G)=\{\ u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}, u_{i6}u_i\}$ for $1\le\ i\le n.$ So $|V(G)|=\ 7n+1$, where $1\le n\le 16$.

Define a labeling function $f: V(H)\to\{1,2,3,\dots,7n+1\}$ as follows,

$$f(u_i)=\begin{cases}7i, & i\equiv 1\ mod(6)\\ 7i-1, & i\equiv 2\ mod(6)\\ 7i-2, & i\equiv 3\ mod(6)\\ 7i-3, & i\equiv 4\ mod(6)\\ 7i-4, & i\equiv 5\ mod(6)\\ 7i-5, & i\equiv 0\ mod(6)\end{cases}$$

$$f(u_{i1})=\begin{cases}7i-5, & i\not\equiv 0\ mod(6)\\ 7i-4, & i\equiv 0\ mod(6)\end{cases}$$

$$f(u_{i2})=\begin{cases}7i-4, & i\not\equiv 0,5\ mod(6)\\ 7i-3, & i\equiv 0,5\ mod(6)\end{cases}$$

$$f(u_{i3})=\begin{cases}7i-3, & i\not\equiv 0,4,5\ mod(6)\\ 7i-2, & i\equiv 0,4,5\ mod(6)\end{cases}$$

$$f(u_{i4})=\begin{cases}7i-2, & i\not\equiv 0,3,4,5\ mod(6)\\ 7i-1, & i\equiv 0,3,4,5\ mod(6)\end{cases}$$

$$f(u_{i5})=\begin{cases}7i-1, & i\equiv 1\ mod(6)\\ 7i, & i\not\equiv 1\ mod(6)\end{cases}$$

Now, consider Greatest Common Devisor of $f(u_i)$ and other four vertices which are $f(u_{i1}), f(u_{i2}), f(u_{i3}), f(u_{i4}), f(u_{i5})$ and $f(u_{i6})$.

Case 1.

Assume $i\equiv 1\ mod(6)$, so that $f(u_i) = 7i$, for all $1\le n\le 16$. Note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i, 7i-5) = 1$(are not multiple of 5 and differ by 5)
$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i, 7i-4) = 1$(odd integers that differ by 4)
$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i, 7i-3) = 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i, 7i-2) = 1$(consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i, 7i-1) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i, 7i+1) = 1$(consecutive positive numbers)

Case 2.

Assume $i\equiv 2\ mod(6)$, so that $f(u_i) = 7i-1$, for all $1\le n\le 16$. Note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i-1, 7i-5) = 1$(odd integers that differ by 4)
$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i-1, 7i-4) = 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i-1, 7i-3) = 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i-1, 7i-2) = 1$(consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i-1, 7i) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i-1, 7i+1) = 1$(consecutive odd numbers)

Case 3.

Assume $i\equiv 3\ mod(6)$, so that $f(u_i) = 7i-2$, for all $1\le n\le 16$. Note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i-2, 7i-5) = 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i-2, 7i-4) = 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i-2, 7i-3) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i-2, 7i-1) = 1$(consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i-2, 7i) = 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i-2, 7i+1) = 1$(are not multiple of 3 and differ by 3)

Case 4.

Assume $i\equiv 4\ mod(6)$, so that $f(u_i) = 7i-3$, for all $1\le n\le 16$. Note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i-3, 7i-5) = 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i-3, 7i-4) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i-3, 7i-2) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i-3, 7i-1) = 1$(consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i-3, 7i) = 1$(are not multiple of 3 and differ by 3)
$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i-3, 7i+1) = 1$(odd numbers that differ by 4)

Case 5.

Assume $i\equiv 5\ mod(6)$, so that $f(u_i) = 7i-4$, for all $1\le n\le 16$. Note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i-4, 7i-5) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i-4, 7i-3) = 1$(consecutive positive numbers)
$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i-4, 7i-2) = 1$(consecutive odd numbers)
$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i-4, 7i-1) = 1$(are not multiple of 3 and differ by 3)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i-4, 7i) = 1$(odd numbers that differ by 4)
$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i-4, 7i+1) = 1$(are not multiple of 5 and differ by 5)

Case 6.

Assume $i \equiv 0\ mod(6)$, so that $f(u_i) = 7i - 5$, for all $1 \leq n \leq 16$. Note that,

$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(7i-5, 7i-4) = 1$(consecutive positive numbers)

$g.c.d(f(u_i), f(u_{i2})) = g.c.d(7i-5, 7i-3) = 1$(consecutive odd numbers)

$g.c.d(f(u_i), f(u_{i3})) = g.c.d(7i-5, 7i-2) = 1$(are not multiple of 3 and differ by 3)

$g.c.d(f(u_i), f(u_{i4})) = g.c.d(7i-5, 7i-1) = 1$(odd numbers that differ by 4)

$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(7i-5, 7i) = 1$(are not multiple of 3 and differ by 3)

$g.c.d(f(u_i), f(u_{i6})) = g.c.d(7i-5, 7i+1) = 1$(odd numbers that are not multiple of 3 and differ by 6)

Clearly, vertex labels are distinct. Thus labeling defined above gives a prime labeling for H. Therefore H is a prime graph.

For example, the prime labeling of the graph obtained by replacing every edge of a star graph $K_{1,5}$ by $K_{2,6}$ using labeling appears in figure 2.

2.3. Theorem 3

The graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,8}$ is a prime graph, where $n \geq 1$.

Proof:

Let H be the graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,8}$ is a prime graph. Let the vertices of $K_{1,n}$ be $u_0, u_1, u_2, \ldots, u_n$ with u_0 as center vertex and it is label as 1.

Furthermore, every edge u_0u_i of $K_{1,n}$ replaced by $u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i6}, u_{i7}$ and u_{i8} by joining $u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}, u_{i6}u_i, u_0u_{i7}, u_{i7}u_i$, u_0u_{i8} and $u_{i8}u_i$, for $1 \leq i \leq n$.

Then the new vertex set is $V(G) = \{u_0, u_i, u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i6}, u_{i7}, u_{i8}\}$ for $1 \leq i \leq n$ and new edge set is,

$E(G) = \{u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}, u_{i6}u_i,$
$u_0u_{i7}, u_{i7}u_i, u_0u_{i8}, u_{i8}u_i\}$ for $1 \leq i \leq n$. So $|V(G)| = 9n+1$, where $n \geq 1$.

Define a labeling function $f: V(H) \rightarrow \{1,2,3, \ldots, 9n+1\}$ as follows,

$$f(u_i) = \begin{cases} 9i-2, & i \equiv 1,5,7,9\ mod(10) \\ 9i-1, & i \equiv 0,2,6,8\ mod(10) \\ 9i-4, & i \equiv 3\ mod(10) \\ 9i-5, & i \equiv 4\ mod(10) \end{cases}$$

$$f(u_{i1}) = 9i - 7\ for\ all\ i$$

$$f(u_{i2}) = 9i - 6\ for\ all\ i$$

$$f(u_{i3}) = \begin{cases} 9i-5, & i \not\equiv 4\ mod(10) \\ 9i-4, & i \equiv 4\ mod(10) \end{cases}$$

$$f(u_{i4}) = \begin{cases} 9i-4, & i \not\equiv 3,4\ mod(10) \\ 9i-3, & i \equiv 3,4\ mod(10) \end{cases}$$

$$f(u_{i5}) = \begin{cases} 9i-3, & i \not\equiv 3,4\ mod(10) \\ 9i-2, & i \equiv 3,4\ mod(10) \end{cases}$$

$$f(u_{i6}) = \begin{cases} 9i-2, & i \equiv 0,2,6,8\ mod(10) \\ 9i-1, & i \not\equiv 0,2,6,8\ mod(10) \end{cases}$$

$$f(u_{i7}) = 9i\ \ for\ all\ i$$

$$f(u_{i8}) = 9i + 1\ \ for\ all\ i$$

Now, consider Greatest Common Divisor of $f(u_i)$ and other four vertices which are $f(u_{i1}), f(u_{i2}), f(u_{i3}), f(u_{i4}), f(u_{i5}), f(u_{i6}), f(u_{i7})$ and $f(u_{i8})$.

Case 1.

Assume $i \equiv 1,5,7,9\ mod(10)$, so that $f(u_i) = 9i - 2$, note that,

$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(9i-2, 9i-7) = 1$ (are not multiple of 5 and differ by 5)

$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(9i-2, 9i-6) = 1$ (are odd numbers that differ by 4)

$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(9i-2, 9i-5) = 1$(are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(9i-2, 9i-4) = 1$ (consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(9i-2, 9i-3) = 1$ (consecutive positive numbers)

$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(9i-2, 9i-1) = 1$ (consecutive positive numbers)

$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(9i-2, 9i) = 1$ (consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(9i-2, 9i+1) = 1$ (are not multiple of 5 and differ by 5)

Case 2.

Assume $i \equiv 0,2,6,8\ mod(10)$, so that $f(u_i) = 9i - 1$, note that,

$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(9i-1, 9i-7) = 1$ (are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(9i-1, 9i-6) = 1$ (are not multiple of 5 and differ by 5)

$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(9i-1, 9i-5) = 1$(are odd numbers that differ by 4)

$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(9i-1, 9i-4) = 1$ (are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(9i-1, 9i-3) = 1$ (consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(9i-1, 9i-2) = 1$ (consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(9i-1, 9i) = 1$ (consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(9i-1, 9i+1) = 1$ (consecutive odd numbers)

Case 3.
Assume $i \equiv 3 \; mod(10)$, so that $f(u_i) = 9i-4$, note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(9i-4, 9i-7) = 1$ (are not multiple of 3 and differ by 3)
$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(9i-4, 9i-6) = 1$ (consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(9i-4, 9i-5) = 1$(consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(9i-4, 9i-3) = 1$ (consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(9i-4, 9i-2) = 1$ (consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(9i-4, 9i-1) = 1$ (are not multiple of 3 and differ by 3)
$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(9i-4, 9i) = 1$ (are not multiple of 2 and differ by 4)
$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(9i-4, 9i+1) = 1$ (are not multiple of 5 and differ by 5)

Case 4.
Assume $i \equiv 3 \; mod(10)$, so that $f(u_i) = 9i-5$, note that,
$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(9i-5, 9i-7) = 1$ (consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(9i-5, 9i-6) = 1$ (consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(9i-5, 9i-4) = 1$(consecutive positive numbers)
$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(9i-5, 9i-3) = 1$ (consecutive odd numbers)
$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(9i-5, 9i-2) = 1$ (are not multiple of 3 and differ by 3)
$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(9i-5, 9i-1) = 1$ (are odd numbers that differ by 4)
$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(9i-5, 9i) = 1$ (are not multiple of 5 and differ by 5)
$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(9i-5, 9i+1) = 1$ (are not multiple of 3 and differ by 6)

Clearly, vertex labels are distinct. Thus labeling defined above gives a prime labeling for H. Therefore H is a prime graph.

For example, the prime labeling of the graph obtained by replacing every edge of a star graph $K_{1,3}$ by $K_{2,8}$ using labeling appears in figure 3.

2.4. Theorem 4

The graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,9}$ is a prime graph, where $n \geq 1$.

Proof:

Let H be the graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,8}$ is a prime graph. Let the vertices of $K_{1,n}$ be $u_0, u_1, u_2, \dots, u_n$ with u_0 as center vertex and it is label as 1.

Furthermore, every edge u_0u_i of $K_{1,n}$ replaced by $u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i6}, u_{i7}, u_{i8}$ and u_{i9} by joining $u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}, u_{i6}u_i, u_0u_{i7}, u_{i7}u_i, \quad u_0u_{i8}, u_{i8}u_i, u_0u_{i9}$ and $u_{i9}u_i$, for $1 \leq i \leq n$.

Then the new vertex set is $V(G) = \{u_0, u_i, u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5}, u_{i6}, u_{i7}, u_{i8}, , u_{i9}\}$ for $1 \leq i \leq n$ and new edge set is,

$E(G) = \{u_0u_{i1}, u_{i1}u_i, u_0u_{i2}, u_{i2}u_i, u_0u_{i3}, u_{i3}u_i, u_0u_{i4}, u_{i4}u_i, u_0u_{i5}, u_{i5}u_i, u_0u_{i6}, u_{i6}u_i, u_0u_{i7}, u_{i7}u_i, u_0u_{i8}, u_{i8}u_i, u_0u_{i9}, u_{i9}u_i\}$ for $1 \leq i \leq n$. So $|V(G)| = 10n+1$, where $n \geq 1$.

Define a labeling function $f: V(H) \to \{1,2,3, \dots, 10n+1\}$ as follows,

$$f(u_i) = \begin{cases} 10i+1, & i \not\equiv 2,5,8 \; mod(9) \\ 10i-3, & i \equiv 2,5,8 \; mod(9) \end{cases}$$

$$f(u_{i1}) = 10i-8 \;\; for \; all \; i$$
$$f(u_{i2}) = 10i-7 \;\; for \; all \; i$$
$$f(u_{i3}) = 10i-6 \;\; for \; all \; i$$
$$f(u_{i4}) = 10i-5 \;\; for \; all \; i$$
$$f(u_{i5}) = 10i-4 \;\; for \; all \; i$$

$$f(u_{i6}) = \begin{cases} 10i-3, & i \not\equiv 2,5,8 \; mod(9) \\ 10i-2, & i \equiv 2,5,8 \; mod(9) \end{cases}$$

$$f(u_{i7}) = \begin{cases} 10i-2, & i \not\equiv 2,5,8 \; mod(9) \\ 10i-1, & i \equiv 2,5,8 \; mod(9) \end{cases}$$

$$f(u_{i8}) = \begin{cases} 10i-1, & i \not\equiv 2,5,8 \; mod(9) \\ 10i, & i \equiv 2,5,8 \; mod(9) \end{cases}$$

$$f(u_{i9}) = \begin{cases} 10i, & i \not\equiv 2,5,8 \; mod(9) \\ 10i+1, & i \equiv 2,5,8 \; mod(9) \end{cases}$$

Now, consider Greatest Common Devisor of $f(u_i)$ and other four vertices which are $f(u_{i1}), f(u_{i2}), f(u_{i3}), f(u_{i4}), f(u_{i5}), f(u_{i6}), f(u_{i7}), f(u_{i8})$, and $f(u_{i9})$.

Case 1.

Assume $i \not\equiv 2,5,8 \ mod(9)$, so that $f(u_i) = 10i + 1$, note that,

$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(10i + 1, 10i - 8) = 1$(are not multiple of 3 and differ by 9)

$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(10i + 1, 10i - 7) = 1$ (are not multiple of 2 and differ by 8)

$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(10i + 1, 10i - 6) = 1$(are not multiple of 7 and differ by 7)

$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(10i + 1, 10i - 5) = 1$ (are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(10i + 1, 10i - 4) = 1$ (are not multiple of 5 and differ by 5)

$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(10i + 1, 10i - 3) = 1$ (odd numbers that differ by 4)

$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(10i + 1, 10i - 2) = 1$ (are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(10i + 1, 10i - 1) = 1$(consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i9})\big) = g.c.d(10i + 1, 10i) = 1$ (consecutive positive numbers)

Case 2.

Assume $i \equiv 2,5,8 \ mod(9)$, so that $f(u_i) = 10i - 3$, note that,

$g.c.d\big(f(u_i), f(u_{i1})\big) = g.c.d(10i - 3, 10i - 8) = 1$ (are not multiple of 5 and differ by 5)

$g.c.d\big(f(u_i), f(u_{i2})\big) = g.c.d(10i - 3, 10i - 7) = 1$ (odd numbers that differ by 4)

$g.c.d\big(f(u_i), f(u_{i3})\big) = g.c.d(10i - 3, 10i - 6) = 1$(are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i4})\big) = g.c.d(10i - 3, 10i - 5) = 1$ (consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i5})\big) = g.c.d(10i - 3, 10i - 4) = 1$ (consecutive positive numbers)

$g.c.d\big(f(u_i), f(u_{i6})\big) = g.c.d(10i - 3, 10i - 2) = 1$ (consecutive positive numbers)

$g.c.d\big(f(u_i), f(u_{i7})\big) = g.c.d(10i - 3, 10i - 1) = 1$ (consecutive odd numbers)

$g.c.d\big(f(u_i), f(u_{i8})\big) = g.c.d(10i - 3, 10i) = 1$(are not multiple of 3 and differ by 3)

$g.c.d\big(f(u_i), f(u_{i9})\big) = g.c.d(10i - 3, 10i + 1) = 1$ (odd numbers that differ by 4)

Clearly, vertex labels are distinct. Thus labeling defined above gives a prime labeling for H. Therefore H is a prime graph.

For example, the prime labeling of the graph obtained by replacing every edge of a star graph $K_{1,3}$ by $K_{2,9}$ using labeling appears in figure 4.

3. Results and Discussion

In general, all the graphs are not prime, it is very interesting to investigate graph families which admit prime labeling. In this work, we prove that the graph obtained by replacing every edge of star graph $K_{1,n}$ by $K_{2,m}$ is prime graph, where $n \geq 1$ for $m = 4, 8, 9$ and $1 \leq n \leq 16$ for $m = 6$.

4. Conclusions

The prime numbers and their behavior are of great importance as prime numbers are scattered and there are arbitrarily large gaps in the sequence of prime numbers. Through this work, we were able to introduce a new type of graphs which have prime labeling. We consider a graph that obtained by replacing every edge of star graphs $K_{1,n}$ by $K_{2,m}$,where $n \geq 1$ for $m = 4, 8, 9$ and $1 \leq n \leq 16$ for $m = 6$.

Determining prime labeling for the graph that obtained by replacing every edge of star graphs $K_{1,n}$ by $K_{2,m}$ more difficult for when $m = 7$ and $m \geq 10$. Also we expect that all graphs of obtained by replacing every edge of star graphs $K_{1,n}$ by $K_{2,m}$ are prime when $n \geq 1$ and $m \geq 2$.

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Figure-1. Prime labeling of the graph obtained by replacing every edge of star graph $K_{1,3}$ by $K_{2,4}$

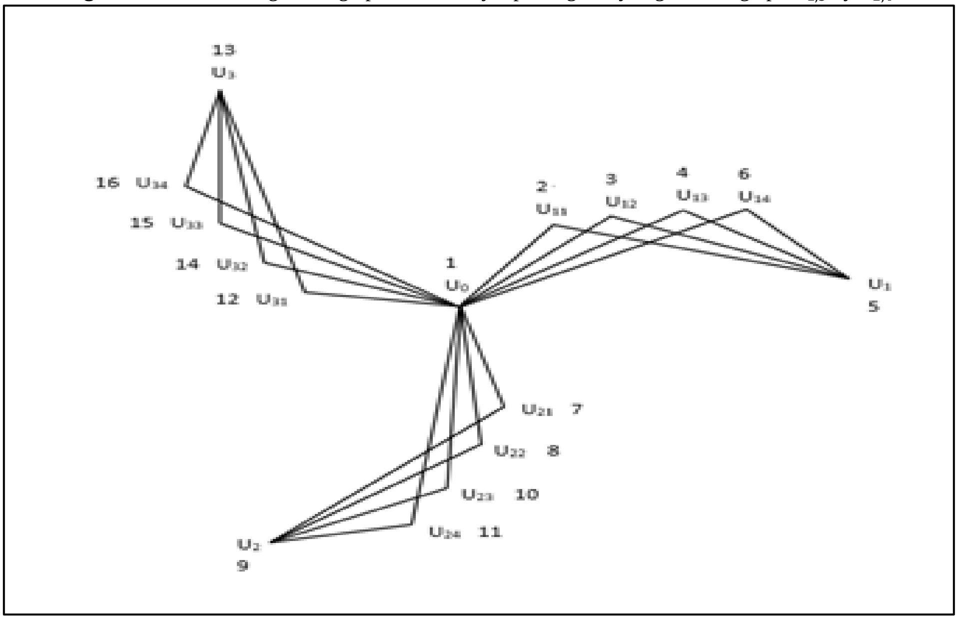

Figure-2. Prime labeling of the graph obtained by replacing every edge of star graph $K_{1,5}$ by $K_{2,6}$

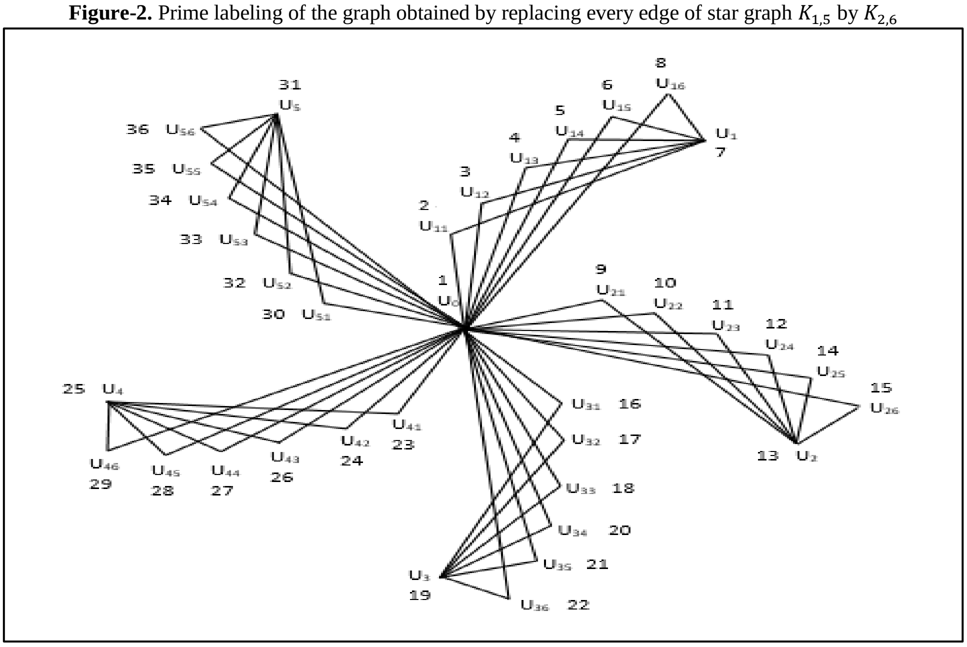

Figure-3. Prime labeling of the graph obtained by replacing every edge of star graph $K_{1,3}$ by $K_{2,8}$

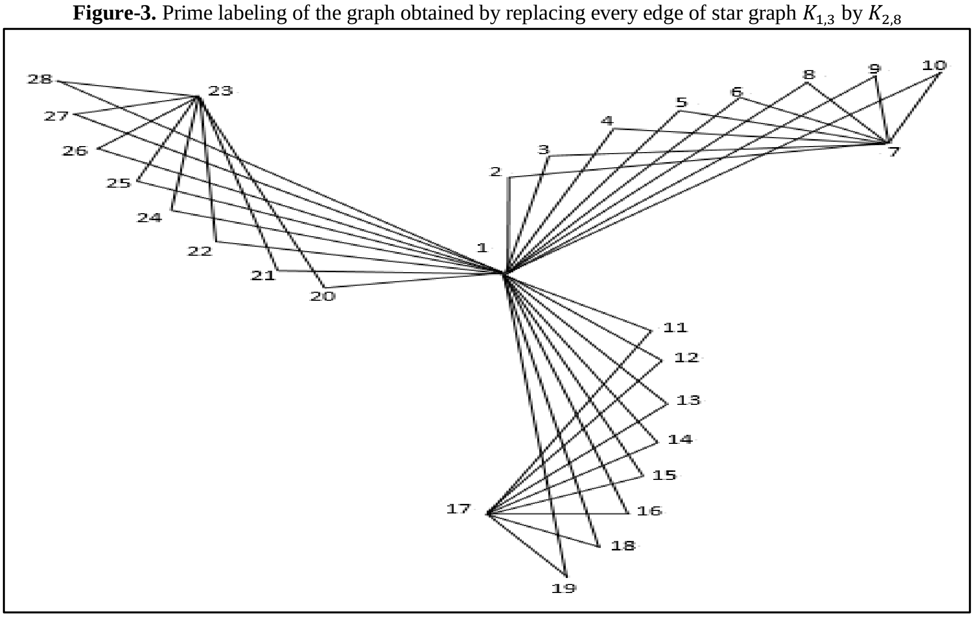

Figure-4. Prime labeling of the graph obtained by replacing every edge of star graph $K_{1,3}$ by $K_{2,9}$