Sumerianz Journal of Scientific Research, 2021, Vol. 4, No. 1, pp. 3-7
ISSN(e): 2617-6955, ISSN(p): 2617-765X
Website: https://www.sumerianz.com
DOI: https://doi.org/10.47752/sjsr.41.3.7

Original Article Open Access

Caristi's Fixed Point Theorem in G-Metric Spaces and Applications

Iluno C.
Department of mathematics and Statistics, Lagos State Polytechnics Ikorodu, Lagos, Nigeria

Adetowubo A.
Department of mathematics and Statistics, Lagos State Polytechnics Ikorodu, Lagos, Nigeria

Adewale O. K. *
Department of mathematics, University of Lagos, Nigeria
Email: adewalekayode2@yahoo.com

Article History
Received: December 17, 2020
Revised: January 25, 2021
Accepted: January 27, 2021
Published: January 29, 2021

Abstract

In this paper, we give some versions of Caristi's fixed point theorem in a more general metric spacesetting. Our work extends a good number of results in this area of research.
Keywords: Caristi's fixed point; G- metric spaces; Complete G- metric spaces; Unique fixed point and cauchy sequence.

1. Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a vital role in mathematics and applied sciences. Several authors in this area of studies tried to generalize the usual notion of metric space (X,d) to a more general setting. Mustafa and Sims with many others introduced a generalized metric space [1-13].

In the last decades, Caristi's fixed point theorem has been generalized and extended in several directions and the related references therein. The following were known definitions and theorem in the literature.

Definition 1.1 [1]. Let X be a non-empty set. Suppose that d: XxX $\to [0,\infty)$ satisfies:

(i) $0 \le d(x,y) \forall x,y . \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x) \ \forall x,y \in X$

2000AMS Subject Classification: 47H10

(iii) $d(x,y) \le d(x,z) + d(z,y) \forall x, y, z \in X$
Then d is called a metric on X and (X, d) is called a metric space.

Definition 1.2 [12]. Let X be a non-empty set and $G: X \times X \times X \to [0,\infty)$ be a function satisfying the following properties;

(i) $G(x,y,z) = 0$ if and only if $x = y = z$,
(ii) $G(x,x,y) > 0 \forall x,y \in X$, with $x \ne y$
(iii) $G(x,x,y) < G(x,y,z) \qquad \forall x,y,z, \in X$, with $z \neq y$
(iv) $G(x,y,z) = G(\rho(x,y,z))$(symmetry) where ρ denotes the permutation functions.
(v) $G(x,y,z) \le G(x,a\,a,) + G(a,y,Z) \forall a,x,y,z \in X$ (rectangle inequality),
then the function G is called a G – metric

Theorem 1.3 [9]

Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping such that

$$d(x,Tx) \le \phi(x) - \phi(Tx)$$

for all $x \in X$, where $\phi: X \to [0,\infty)$is a lower semi continuous mapping. Then T has at least a fixed point.

Theorem 1.4 [9]

Let (X, d) be a complete metric space and let $T: X \to CB(X)$be a mapping such that
$H(Tx,Ty) \le \eta(d(x, y))\, d(x, y)$
for all$x, y \in X$, where $\eta : (0,\infty) \to [0,1)$is a mapping such that $LimSup_{r \to x}\eta(r)$<1, for all $r \in [0,\infty)$. Then T has a fixed point.

Theorem 1.5 [9]

Let (X, d) be a complete metric space, and let $T: X \to X$be a mapping such that

$$d(x,y) \le \phi(x,y) - \phi(Tx,Ty)$$

for all $x,y \in X$, where $\phi: X \to [0,\infty)$is lower semicontinuous with respect to the first variable. Then T has a unique fixed point.

Theorem 1.6 [9]

Let (X,d)be a complete metric space and let $T: X \to X$ be a mapping such that for some $a \in [0,1)$

$$d(Tx,Ty) \le ad(x,y,)$$

for all $x,y \in X$. Then T has a unique fixed point.

*Corresponding Author

Problem 1.7

Let (X,G) be a complete G-metric space and let $T: X \to CB(X)$ be a multivalued mapping such that

$$H(Tx,Ty,Tz) \le \mu G(x,y,z)$$

for all $x,y,z \in X$where $\mu: R^+ \to R^+$ is continuous and increasing map such that $\mu(t) < t$, for all $t > 0$.Does Thave a fixed point?

Problem 1.8

Let (X,G) be a complete G - metric space and let $T: X \to CB(X)$ be a mapping such that

$$H(Tx,Ty,Tz) \le \eta\big(G(x,y,z)\big)G(x,y,z)$$

for all $x,y,z \in X$, where $\eta: [0,\infty) \to [0,1)$ is a mapping such that $\mathrm{LimSup}_{r\to t^+}\eta(r) < 1$, for all $r \in (0,+\infty)$. Does Thave a fixed point?

Theorem 1.9 [9]

Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping such that

$$d(Tx,Ty) \le \eta(d(x,y))$$

where $\eta: [0,\infty) \to [0,\infty)$is a lower semi continuous mapping such that $\eta(t) < t$, for each $t > 0$, and $\frac{\eta(t)}{t}$ is a non decreasing map. Then T has a unique fixed point.

Theorem 1.10 [9]

Let$S: B(Z) \to B(Z)$be an upper semi continuous operator and assume that the following conditions are satisfied:

(i) $f: Z \times T \to R$ and $\vartheta: Z \times T \times R \to R$are continuous and bounded;

(ii) for all $h,k \in B(Z)$, if

$0 < d(h,k) < 1$implies $|\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \le \frac{1}{2}d^2(h,k)$,

and

$d(h,k) \ge 1$implies $|\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \le \frac{2}{3}d(h,k)$

where $x \in Z$ and $y \in \tau$. Then the functional equation (a) has a bounded solution.

2. Main Result

Theorem 2.1

Let (X, G) be a complete G-metric space, and $T: X \to X$, a mapping such that

$$G(x,x,y) \le \phi(x,y,y) - \phi(Tx,Ty,Ty)$$

for all $x,y \in X$, where $\phi: X \to [0,\infty)$ is lower semi continuous with respect to the first variable. Then T has a unique fixed point.

Proof:

For each$x \in X$, let $y = Tx$ and $\psi(x) = \phi(x,Tx,Tx)$. Then for each $x \in X$

$$G(x,x,Tx) \le \psi(x) - \psi(Tx)$$

and ψ is a lower semi continuous mapping. Thus, setting $G(x,x,Tx) = d(x,Tx)$ and applying Theorem 1.3, we obtain the desired result.

To see the uniqueness of the fixed point suppose u and v are two distinct fixed points for T. Then

$$G(u,u,v) \le \phi(u,v,v) - \phi(Tu,Tv,Tv) = \phi(u,v,v) - \phi(u,v,v) = 0$$

Thus, $u = v$.

Theorem 2.2

Let (X,G) be a complete G - metric space and $T: X \to X$, a mapping such that for some $a \in [0,1)$

$$G(Tx,Ty,Tz) \le aG(x,y,z) \quad (1)$$

for all $x,y,z \in X$. Then T has a unique fixed point.

Proof:

Define $\phi(x,y,z) = \frac{G(x,y,z)}{1-a}$.

Equation 1 shows that

$$(1-a)G(x,y,z) \le G(x,y,z) - G(Tx,Ty,Tz).$$

That is

$$G(x,y,z) \le \frac{G(x,y,z)}{1-a} - \frac{G(Tx,Ty,Tz)}{1-a}$$

and so

$$G(x,y,z) \le \phi(x,y,z) - \phi(Tx,Ty,Tz)$$

applying theorem 2.1, one can conclude that Thas a unique fixed point.

Theorem 2.3

Let (X, G) be a complete G- metric space and let $T: X \to X$ be a mapping such that

$$G(Tx,Ty.Tz) \le \eta(G(x,y,z)) \quad (2)$$

where $\eta: [0,\infty) \to [0,\infty)$ is a lower semi continuous mapping such that $\eta(t) < t$, for each $t > 0$, and $\frac{\eta(t)}{t}$ is a non decreasing map. Then T has a unique fixed point.

Proof:

Define $\phi(x,y,z) = \frac{G(x,y,z)}{1-\frac{\eta(G(x,y,z))}{G(x,y,z)}}$, if $x \ne y$and otherwise $(\phi(x,x,x)) = 0$. Then equation 2 shows that

$$\left(1-\frac{\eta(G(x,y,z))}{G(x,y,z)}\right)G(x,y,z) \le G(x,y,z) - G(Tx,Ty,Tz)$$

It means that

$$G(x,y,z) \le \frac{G(x,y,z)}{\left(1-\frac{\eta(G(x,y,z))}{G(x,y,z)}\right)} - \frac{G(Tx,Ty,Tz)}{\left(1-\frac{\eta(G(x,y,z))}{G(x,y,z)}\right)}$$

Since $\frac{\eta(t)}{t}$ is non decreasing and $G(Tx,Ty,Tz) < G(x,y,z)$,

$$G(x,y,z) \le \frac{G(x,y,z)}{\left(1-\frac{\eta(G(x,y,z))}{G(x,y,z)}\right)} - \frac{G(Tx,Ty,Tz)}{\left(1-\frac{\eta(G(x,y,z))}{G(x,y,z)}\right)} = \phi(x,y,z) - \phi(Tx,Ty,Tz)$$

and so by applying theorem 2.1, one can conclude that T has a unique fixed point.

The following results are the main results of this paper and play a crucial role to find the partial answers for Problem 1.7 and Problem 1.8. Compare to the work of Farshid Khojasteh, Erdal Karapinar and Hassan Khandani dealing with distance in a straight line, our work considers distance round a triangle.

Theorem 2.4

Let (X, G) be a complete G - metric space, and let $T: X \to CB(X)$ be a non expansive mapping such that, for each $x \in X$, and for all $y \in Tx$, there exists $z \in Ty$ such that

$$G(x,y,y) \le \phi(x,y,y) - \phi(y,z,z) \qquad 3$$

where $\phi : X \times X \to [0,\infty)$ is lower semi continuous with respect to the first variable. Then T has a fixed point.

Proof:

Let $x_0 \in X$and $x_1 \in Tx_0$. If $x_0 = x_1$ then x_0 is a fixed point and we are through. Otherwise, let $x_0 \ne x_1$. By assumption there exists $x_2 \in Tx_1$ such that

$$G(x_0,x_1,x_1) \le \phi(x_0,x_1,x_1) - \phi(x_1,x_2,x_2)$$

Alternatively, one can choose $x_n \in Tx_{n-1}$, such that $x_n \ne x_{n-1}$ and find $x_{n+1} \in Tx_n$such that

$$0 < G(x_{n-1},x_n,x_n) \le \phi(x_{n-1},x_n,x_n) - \phi(x_n,x_{n+1},x_{n+1}) \qquad 4$$

which means that $[\phi(x_{n-1},x_n,x_n)]_n$ is a non-increasing sequence, bounded below, so it converges to some r ≥ 0. By taking the limit on both sides of equation 4 we have $lim_{n\to\infty} G(x_{n-1},x_n,x_n) = 0$. Also, for all $m,n \in N$ with $m > n$,

$$G(x_n,x_m,x_m) \le \sum_{i=n+1}^{m} G(x_{i-1},x_i,x_i) \le \sum_{i=n+1}^{m} (\phi(x_{i-1},x_i,x_i) - \phi(x_i,x_{i+1},x_{i+1})$$

$$\le \phi(x_n,x_{n-1},x_{n-1}) - \phi(x_m,x_{m+1},x_{m+1}) \qquad 5$$

Therefore, by taking the $LimSup$ on both sides of equation 5, we have

$$\lim_{n\to\infty} Sup(G(x_n,x_m,x_m)) = 0$$

It means that (x_n) is a Cauchy sequence and so it converges to $v \in X$. Now we show that v is a fixed point of T. We have

$$G(v,Tv,Tv) \le G(v,x_n,x_n) + G(x_n,Tv,Tv)$$
$$= G(v,x_n,x_n) + H(Tx_n,Tv,Tv)$$
$$\le G(v,x_n,x_n) + G(x_n,v,v) \qquad 6$$

By taking the limit on both sides of equation 6, we get $G(x,x,Tx) = 0$ and this means that $x \in Tx$.

Theorem 2.5

Let(X,G)be a complete G - metric space, and $T: X \to CB(X)$ be a multivalued function such that

$$H(Tx,Ty,Ty) \le \eta G(x,y,y)$$

for all $x,y \in X$ where $\eta: [0,\infty) \to [0,\infty)$is a lower semicontinuous map such that $\eta(t) < t$ for all $t \in (0,+\infty)$, and $\frac{\eta(t)}{t}$ is nondecreasing. Then T has a fixed point.

Proof:

Let $x \in X$ and $y \in Tx$ then T has a fixed point and the proof is complete, so we suppose that $x \ne y$ Define $\theta(t) = \frac{\eta(t)+t}{2}$ for all $t \in (0,\infty)$. We have

$$H(Tx,Ty,Ty) \le \eta(G(x,y,y)) < \theta(G(x,y,y)) < G(x,y,y)$$

Thus, there exists $\epsilon_0 > 0$ such that $\theta(G(x,y,y)) = H(Tx,Ty,Ty) + \epsilon_0$. So there exists $z \in Ty$ such that $G(y,z,z) < H(Tx,Ty,Ty) + \epsilon_0 = \theta\big(G(x,y,y)\big) < G(x,y,y)$

We again suppose that $z \ne y$; therefore $G(x,y,y) - \theta(d(x,y,y)) \le G(x,y,y) - G(y,z,z)$or equivalently

$$G(x,y,y) < \frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}} + \frac{G(y.z.z)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}$$

Since $\frac{\eta(t)}{t}$ is also a nondecreasing function and $G(y,z,z) < G(x,y,y)$ we get

$$G(x,y,y) < \frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}} + \frac{G(y.z.z)}{1-\frac{\theta(G(y,z,z))}{G(y,z,z)}}$$

Define $\phi(x,y,y) = \frac{G(x,y,y)}{1-\frac{\theta(G(x,y,y))}{G(x,y,y)}}$ if $x \neq y$, otherwise 0 for all $x, y \in X$. It means that

$$G(x,y,y) = \phi(x,y,y) - \phi(y,z,z)$$

Therefore, T satisfies equation 3 of Theorem 2.3 and so we conclude that T has a unique fixed point u and the proof is completed.

2.1. Existence of Bounded Solutions of Functional Equations [9]

Mathematical optimization is one of the fields in which the methods of fixed point theory are widely used. It is well known that dynamic programming provides useful tools for mathematical optimization and computer programming. In this setting, the problem of dynamic programming related to a multistage process reduces to solving the functional equation

$$p(x) = sup_{y\in\tau}\left\{f\,(x,y) + \vartheta\left(x,y,p(\eta(x,y))\right)\right\}, x \in Z \qquad 7$$

where $\eta : Z \times \tau \to Z, f : Z \times \tau \to R$ and $\vartheta: Z \times \tau \times R \to R$. We assume that M and N are Banach spaces, Z ⊂M is a state space, and T ⊂N is a decision space. The studied process consists of a state space, which is the set of the initial state, actions, and a transition model of the process, and a decision space, which is the set of possible actions that are allowed for the process.

Here, we study the existence of the bounded solution of the functional equation. Let B(Z) denote the set of all bounded real-valued functions on W and, for an arbitrary h ∈B(Z), define $\|h\| = Sup_{x\in Z}[h(x)]$. Clearly, $B(W)$, ‖. ‖)) endowed with the metric d defined by

$$d(h,k) = Sup_{x\in Z}|h(x) - k(x)| \qquad 8$$

for all h, k ∈B(Z), is a Banach space. Indeed, the convergence in the space B(Z) with respect to ‖. ‖is uniform. Thus, if we consider a Cauchy sequence $\{h_n\}$ in B(Z), then $\{h_n\}$ converges uniformly to a function, say h_*, that is bounded and so h ∈B(Z). We also define $S : B(Z) \to B(Z)$ by

$$Sh(x) = sup_{y\in\tau}\left\{f\,(x,y) + \vartheta\left(x,y,h(\eta(x,y))\right)\right\} \qquad 9$$

for all h ∈B(Z) and x ∈Z.

Remark: We can extend this toG-metric spaces.

$G(h,k,k) = Sup_{x\in Z}|h(x) - k(x)|$ for all h, k ∈B(Z)

We will prove the following theorem.

Theorem 2.6 Let$S : B(Z) \to B(Z)$be an upper semicontinuous operator defined by (c) and assume that the following conditions are satisfied:

(i) $f : Z \times T \to R$ and $\vartheta: Z \times T \times R \to R$are continuous and bounded;

(ii) for all $h, k \in B(Z)$, if

$$0 < G(h,k,k) < 1 \text{implies } |\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \leq \frac{1}{2}G^2(h,k,k),$$

$$G(h,k,k) \geq 1 \text{implies } |\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \leq \frac{2}{3}G(h,k,k) \qquad 10$$

where $x \in Z$ and $y \in \tau$. Then the functional equation 7 has a bounded solution.

Proof:

Note that (B(Z), d) is a complete G-metric space, where G is the G-metric define by $G(h,k,k) = Sup_{x\in Z}|h(x) - k(x)|$.

Let μ be an arbitrary positive number, x ∈Z, and $h_1, h_2 \in B(Z)$, then there exist y_1,$y_2 \in \tau$such that

$$S(h_1)(x) < f(x,y_1) + \vartheta\left(x,y_1,h_1(\eta(x,y_1))\right) + \mu \qquad 11$$

$$S(h_2)(x) < f(x,y_2) + \vartheta\left(x,y_2,h_2(\eta(x,y_2))\right) + \mu \qquad 12$$

$$S(h_1)(x) \geq f(x,y_1) + \vartheta\left(x,y_1,h_1(\eta(x,y_1))\right) \qquad 13$$

$$S(h_2)(x) \geq f(x,y_2) + \vartheta\left(x,y_2,h_2(\eta(x,y_2))\right) \qquad 14$$

Let ∂: $[0,\infty) \to [0,\infty)$ be defined by

$$\partial(t) = \begin{cases} \frac{1}{2}t^2, 0 < t < 1 \\ \frac{1}{2}t, \qquad t \geq 1 \end{cases}$$

Then we can say that equation 10 is equivalent to

$$|\vartheta(x,y,h(x)) - \vartheta(x,y,k(x))| \leq \partial(G(h,k.k)), \qquad 15$$

for all $h, k \in B(Z$. It is easy to see that $\partial(t) < t, for\ all\ t > 0\ and\ \frac{\partial(t)}{t}$is a nondecreasing function.

Therefore, by using equations 11, 14 and 15, it follows that

$$S(h_1)(x) - S(h_2)(x) < \vartheta\left(x,y_1,h_1(\eta(x,y_1))\right) - \vartheta\left(x,y_2,h_2(\eta(x,y_2))\right) + \mu$$

$$\leq \left|\vartheta\left(x,y_1,h_1(\eta(x,y_1))\right) - \vartheta\left(x,y_2,h_2(\eta(x,y_2))\right)\right| + \mu \leq \partial(G(h_1,h_2,h_2)) + \mu$$

Then we get

$$S(h_1)(x) - S(h_2)(x) < \partial(G(h_1,h_{2,}h_2)) + \mu \qquad 16$$

Analogously, by using equation 12 and 13, we have

$$S(h_2)(x) - S(h_1)(x) < \partial(G(h_1,h_2,h_2)) + \mu \qquad 17$$

Hence, from equation 16 and 17 we obtain

$$|S(h_1)(x) - S(h_2)(x)| < \partial(G(h_1, h_2, h_2) + \mu$$

that is,

$$d\big(S(h_1), S(h_2)\big) < \partial(G(h_1, h_2, h_2)) + \mu$$

Since the above inequality does not depend on $x \in Z$, $\mu > 0$ is taken arbitrary, we conclude immediately that

$$d\big(S(h_1), S(h_2)\big) \leq \partial(G(h_1, h_2, h_2))$$

so we deduce that the operator S is a ∂ - contraction. Thus, due to the continuity of S,
Theorem 2.4 applies to the operator S, which has a fixed point $h^* \in B(Z)$, that is, $h^*$is a bounded solution of the functional equation 7.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors have made equal contributions.

Acknowledgements

The authors would like to acknowledge all the authors in the references for their valuable contributions.

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