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A short note on numbers of the form $k+F_n$, where $k \in \{2,4,8\}$ and F_n is a Fermat number

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Abstract

A *Fermat number* is a number of the form $F_n = 2^{2^n} + 1$, where *n* is an integer ≥ 0. A *Fermat composite* (see Dickson [1] or Hardy and Wright [2] or Ikorong [3]) is a non-prime Fermat number and a *Fermat prime* is a prime Fermat number. Fermat composites and Fermat primes are characterized via divisibility in Ikorong [3] and in Ikorong [4]. It is known (see Ikorong [3]) that for every $j \in \{0, 1, 2, 3, 4\}$, F_j is a Fermat prime and it is also known (see Hardy and Wright [2] or Paul [5]) that F_5 and F_6 are Fermat composites ($F_5 = 2^{2^5} + 1 = 641 \times 6700417$, and since 2013, it is known that $F_{2747497} = 2^{2^{2747497}}$ +1 is Fermat composite number). In this paper, we show [via elementary arithmetic congruences] the following Result (**E.**). *For every integer* $n > 0$ *such that* $n \equiv 1$ *mod* [2], *we have* $F_n - 1 \equiv 4$ *mod* [7]*; and for every integer* $n \geq 2$, *we have* $F_n - 1 \equiv 1$ *mod*[*j*], *where* $j \in \{3, 5\}$. Result **(E.)** immediately implies that there are infinitely many composite numbers of the form $2 + F_n$. Result **(E.)** also implies that the only prime of the form $4 + F_n$ is 7 and the only primes of the form $8 + F_n$ are twin primes 11 and 13. That being said, using result **(E.)** and a special case of a Theorem of Dirichlet on arithmetic progression, we conjecture that there are infinitely many primes of the form $2+F_n$.

Keywords: Fermat numbers; F_n.

1. Introduction

1.1. Theorem A.

(E.). *For every integer* $n > 0$ *such that* $n \equiv 1$ mod [2], *we have* $F_n - 1 \equiv 4$ *mod* [7]; *and for every integer* $n \geq 2$, *we have* $F_n - 1 \equiv 1$ *mod*[*j*]*, where* $j \in \{3, 5\}$.

(E.1). *There are infinitely many composite numbers of the form* $2+F_n$.

(E.2). *The only prime of the form* $4+F_n$ *is* 7 *and the only primes of the form* $8+F_n$ *are twin primes* 11 *and* 13.

To prove Theorem A, we need the following remarks.

Remark 1.0.

Let *n be an integer* > 1 *such that* $n \equiv 1$ *mod* [2]. *If* $2^{2^{n-2}} \equiv 4$ *mod* [7], *then* $2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \equiv 4$ *mod* [7].

(**Proof**. Immediate [indeed observe (via elementary arithmetic congruences) that $2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \equiv 256$ *mod* [7] (since $2^{2^{n-2}} \equiv 4$ *mod* [7]) and $256 \equiv 4$ *mod* [7]]. □)

Proposition 1.1.

Let n be an integer > 0 *such that* $n \equiv 1$ *mod* [2]; *then* $2^{2^n} \equiv 4$ *mod* [7].

(**Proof**. Otherwise

Let n be minimum such that $2^{2^n} \not\equiv 4$ *mod* [7] ($n \equiv 1$ *mod* [2] *and* $n > 0$) (1.1)

Clearly

$$n \geq 3 \quad (1.2)$$

(*since* $2^{2^1} = 4$ and $4 \equiv 4$ *mod*[7]). It is immediate to see that

$$2^{2^n} = 2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \quad (1.3)$$

Now using equality *(1.3)* and inequality *(1.2)*, we easily deduce that *(1.1)* clearly implies that

$2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \times 2^{2^{n-2}} \not\equiv 4$ *mod* [7]; $2^{2^{n-2}} \equiv 4$ *mod* [7] ($n \equiv 1$ *mod* [2] and $n>1$) (1.4)

(1.4) clearly contradicts Remark 1.0. □).

Remark 1.2.

Let n be an integer ≥ 3. *If* $2^{2^{n-1}} \equiv 1$ *mod[j] where* $j \in \{3, 5\}$, *then* $2^{2^{n-1}} \times 2^{2^{n-1}} \equiv 1$ *mod[j]*.

(**Proof**. Immediate [via elementary arithmetic congruences]. □)

Proposition 1.3.

Let n be an integer ≥ 2; *then* $2^{2^n} \equiv 1$ *mod[j], where* $j \in \{3, 5\}$.

(**Proof.** Otherwise

Let n be minimum such that $2^{2^n} \not\equiv 1$ *mod[j] where* $j \in \{3, 5\}$ (1.5)

Clearly

$$n \geq 3 \quad (1.6)$$

(since $2^{2^2} = 16$ and $16 \equiv 1$ *mod[j]* where $j \in \{3, 5\}$). It is immediate to see that

$$2^{2^n} = 2^{2^{n-1}} \times 2^{2^{n-1}} \quad (1.7)$$

Now using equality *(1.7)* and inequality *(1.6)*, we easily deduce that *(1.5)* clearly implies that

$2^{2^{n-1}} \times 2^{2^{n-1}} \not\equiv 1$ *mod[j], where* $j \in \{3, 5\}$ *and where* $2^{2^{n-1}} \equiv 1$ *mod[j];* $n \geq 3$ (1.8)

(1.8) clearly contradicts Remark 1.2. □).

Proposition 1.4.

Let n be an integer ≥ 3 *such that* $n \equiv 1$ *mod*[2]. *Then* $2 + F_n \equiv 0$ *mod*[7] and $2 + F_n$ *is composite*.

(**Proof.** Clearly

$$2 + (2^{2^n} + 1) \equiv 0 \text{ mod } [7] \quad (1.9)$$

[Observe that $2^{2^n} \equiv 4$ *mod* [7] (use Proposition 1.1) and use elementary arithmetic congru-ences]. So $2 + F_n \equiv 0$ *mod*[7] and $2 + F_n$ is composite [use congruence *(1.9)* and observe that $2 + (2^{2^n} + 1) = 2+F_n$ and $2+Fn > 7$ (note that $n \geq 3$)]. Proposition 1.4 immediately follows). □

Proposition 1.5.

Let n be an integer ≥ 2. *Then* $4 + F_n \equiv 0$ *mod*[3] *and* $4 + F_n$ *is composite*.

(**Proof.** Clearly

$$4 + (2^{2^n} + 1) \equiv 0 \textit{ mod } [3] \quad (1.10)$$

[Observe that $2^{2^n} \equiv 1$ *mod* [3] (use Proposition 1.3) and use elementary arithmetic congru-ences]. So $4 + F_n \equiv 0$ *mod*[3] and $4 + F_n$ is composite [use congruence *(1.10)* and observe that $4 + (2^{2^n} + 1) = 4+Fn$ and $4+Fn > 3$ (note that $n \geq 2$)]. Proposition 1.5 immediately follows). □

Proposition 1.6.

Let n be an integer ≥ 2. *Then* $8 + F_n \equiv 0$ *mod*[5] and $8 + F_n$ *is composite*.

(**Proof.** Clearly

$$8 + (2^{2^n} + 1) \equiv 0 \textit{ mod } [5] \quad (1.11)$$

[Observe that $2^{2^n} \equiv 1$ *mod* [5] (use Proposition 1.3) and use elementary arithmetic congru-ences]. So $8 + F_n \equiv 0$ mod[5] and $8 + F_n$ is composite [use congruence *(1.11)* and observe that $8 + (2^{2^n} + 1) = 8+Fn$ and $8+Fn > 5$ (note that $n \geq 2$)]. Proposition 1.6 immediately follows). □

Remark 1.7.

There are infinitely many composite numbers of the form $2 + F_n$ *or there are infinitely many prime numbers of the form* $2 + F_n$.

(**Proof.** Immediate). □

Having made the previous Remarks and Propositions, then Theorem A becomes immediate to prove.

Proof of Theorem A.

(**E.**). Immediate [use Proposition 1.1 and Proposition 1.3 and observe that $2^{2^n} = F_n - 1$].

(**E.1**). Immediate [use Proposition 1.4 and Remark 1.7].

(**E.2**). Clearly the only prime of the form $4 + F_n$ is 7 [observe that $4 + F_0 = 7$ and $4 + F_1 = 9$ and use Proposition 1.5]; and clearly the only primes of the form $8 + F_n$ are twin primes 11 and 13 [observe that $8 + F_0 = 11$ and $8+F_1=13$ and use Proposition 1.6]. □

Now usig Result (E.) of Theorem *A*, then we end this paper by looking at numbers of the form $2+F_n$ which are prime [observe that for every $n \in \{0, 1, 2, 4\}$, $2 + F_n$ is prime and $2 + F_3$ is not prime].

Observation. *It is natural to conjecture that there are infinitely many primes of the form* $2 + F_n$.

Indeed observing [via Result (E.) of Theorem *A*] that

$$F_n - 1 \equiv 1 \ mod\ [3] \ and\ F_n - 1 \equiv 1 \ mod\ [5] \ for\ every\ integer\ n \geq 2 \qquad (1.12),$$

clearly

$$2 + F_n \equiv 4 \ mod\ [5] \ and\ 2 + F_n \equiv 1 \ mod[3] \ for\ every\ integer\ n \geq 2 \qquad (1.13)$$

[use *(1.12)* and elementary arithmetic congruences].

Now let $A_{4.5} = \{e;\ e\ is\ prime\ and\ e \equiv 4\ mod\ [5]\}$ and let $A_{1.3} = \{e';\ e'\ is\ prime\ and\ e' \equiv 1\ mod\ [3]\}$. Since it is immediate that

$$(4, 5) = 1 \ and\ (1, 3) = 1 \qquad (1.14)$$

Then using *(1.14)* coupled with a special case of a Theorem of Dirichlet on arithmetic progression [we recall that *Theorem of Dirichlet on arithmetic progression states that For any two positive coprime integers a and d, there are infinitely many primes of the form a + nd, where n is a positive integer (In other words, there are infinitely many primes that are congruent to a modulo d*)], it follows that

$$card\ (A_{4.5})\ is\ infinite\ and\ card\ (A_{1.3})\ is\ infinite \qquad (1.15)$$

Now using *(1.13)* and *(1.15)*, then it becomes naturel to conjecture the following.

Conjecture. $A_{4.5}$ or $A_{1.3}$ *contains infinitely many numbers of the form* $2+F_n$ ($A_{4.5}$ and $A_{1.3}$ *are defined via the Observation placed just above*).

The previous conjecture immediately implies that there are infinitely many primes of the form $2+F_n$.

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