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Nonexistence of Twins' Paradox in Polygonal and Circular Paths: The Proof of the Real Solution

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Abstract

The so-called Twins' Paradox has been a controversial issue for more than a century, concerning the foundations of modern physics, in this case, the credibility of special relativity itself. In fact, how can a supposed credible theory predict paradoxes, if one assumes that there can be no paradoxes in Nature? Since Einstein's founding paper, the phenomenon of time dilation has been mistakenly used to construct a seemingly insoluble contradiction at the heart of the theory, calling into question the fundamental equivalence between inertial frames of coordinates. In an early paper on the subject, I proved that, whether on a one-way trip or a linear round trip, there is no paradox at all: twins meet again at the same age, and this despite time-dilation. Regarding the round trip, a new dilation factor emerged, in an equation for the final time that is precisely the same for the reference frame of each twin. In a second paper, by means of a well-founded conjecture, these conclusions were generalized to closed regular polygonal paths and, in the limit, to circular paths. A generalized equation was proposed and thoroughly verified, while various aspects of the problem were addressed and highlighted. However, it was only a conjecture, formal proof was missing. This paper brings that proof, thus turning the conjecture into a theorem, i.e., a scientific fact.
Keywords: Special relativity; Twin's paradox; Time dilation; Delta factor; Euclidean and relativistic geometry.

1. Introduction

The puzzling Twins' Paradox, formulated as such by Paul Langevin in Paul [1], is a sort of metaphorical approach to a short statement proposed by Einstein as a "typical consequence" in his founding paper on special relativity (SR) [2]. It remains a very important issue, like a thorn in the heart of a beautiful and fruitful theory, dealing with a somehow tricky phenomenon – time dilation – mainly because of a contradiction that is difficult to resolve. In short: one of two twins remains on Earth while the other gets on a rocket, travels with a velocity near to c and gets back; since the theory consistently sustains that a moving clock runs slower than a 'stationary' one, the voyager should return younger than his twin, when they meet again. The well-known paradox comes from the fact that, also consistently, SR takes all inertial frames as equivalent. Therefore, from the point of view of the traveling twin, it is the other who is moving, and the age discrepancy should be reversed. This is, of course, an absurd.

Although acceleration is required to reach a certain velocity or to reverse the motion, it should be emphasized that, at its base, the issue has nothing to do with accelerations: it results strictly from Lorentz transformations, that is, from SR itself. I examined this issue in Luís [3], including some background revision, since Langevin's and Max von Laue's attempts [4] to modern texts and even videos on YouTube. All of these analyses somehow miss the point; Sooner or later, the explanations become obscure and unsatisfactory. In other words, there is no explanation at all.

In my own analysis in Luís [3] concerning the linear round trip, I began exposing the flaws in the usual premises, and showed that the key for a real solution lies in relativistic asynchrony. In the end, there is, in fact, a time dilation, but it concerns both twins equally, the dilation factor being, not the classical $\gamma = 1/\sqrt{1-\beta^2}$, but the delta factor $\delta = \frac{1}{2}(\gamma^2 + 1)$.

Let us consider a moving observer (twin B) along the x-axis of the inertial frame S of his twin A, supposed at rest, according to standard Lorentz transformations. The gamma factor appears in the first leg of the linear round trip, from the starting point to the returning point, where a third twin, C, stands still. However, when B leaves the starting point, at the instant $t'_0 = 0$, for him *his twin C is in the past*. Because of this delay, when B meets C, despite the time dilation phenomenon, both twins have aged the same, that is $t' = t\sqrt{1-\beta^2}$. In fact, the negative time lag at the moment of departure exactly counterbalances the longer time t for B to reach C, in this one's or A's proper frame, allowing us to understand the coherent physical result of the experiment (due to the equivalence between the frames of B and C).

Most astounding is the following equation, deduced in the paper, for the time required for B and A to meet again at the end of the round trip, which is the same for both twins:

$$T' = T = t\left(\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}\right). \qquad (1)$$

This outcome finally resolves a false paradox, which has persisted for more than a century, and demonstrates once again the amazing consistency of SR.

However, in the sequence of this paper, I was intrigued by the problem of the observer B following a closed polygonal or circular path, also suggested by Einstein (who drew a false common-sense conclusion from it). A strong belief in SR theory leaded led me to the conviction that, in the end, once again both twins should meet at the same age. Then, by a sort of logical imperative, I formulated in Luís [5] the conjecture that the total time, for a regular polygon of n sides, should be given by the following equation:

$$\mathbf{T'} = \mathbf{T} = \frac{n}{2}\ t\left(\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}\right) = \delta_0\, nt, \ (2)$$

Thus preserving what I called the ***delta zero factor*** (related to time t, measured in the frame 'at rest'):

$$\delta_0 = \frac{1}{2}\left(\gamma + \frac{1}{\gamma}\right), \quad (3)$$

Where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the well-known Lorentz factor. One may also write

$$\delta_0 = \frac{1-\beta^2/2}{\sqrt{1-\beta^2}}. \qquad (4)$$

Besides, since $t = \gamma t'$, one may express $\mathbf{T'} = \mathbf{T}$ given by equation (4) as a function of the proper time t' of the moving observer B:

$$\mathbf{T'} = \mathbf{T} = \delta\, nt', \quad \text{where} \quad \delta = \gamma\delta_0 = \frac{1}{2}(\gamma^2 + 1) = \frac{1-\beta^2/2}{1-\beta^2} \qquad (5)$$

is the ***delta factor***.

I have shown in Luís [5] that this conjecture appears to be true; at least, it proved to be true for polygons from three to ten, and even twenty, sides. This cannot be a coincidence. However, as stated above, there was still no definitive mathematical proof.

2. The Proof

2.1. Prerequisites

One can find in Luís [5] the fundamental ideas and concepts, especially mathematical formalism, necessary to achieve our goal. Here only those indispensable to follow the exposition are resumed, as well as Figure 1, which makes it easier to understand what follows.

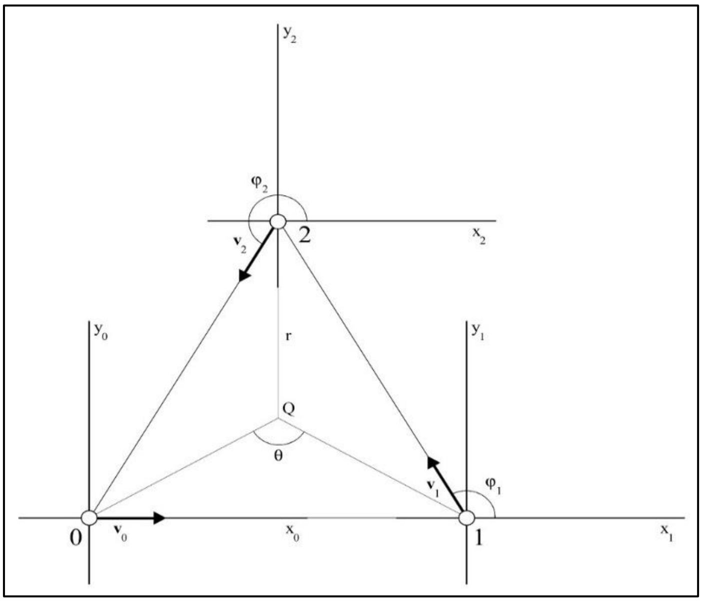

Figure-1. An equilateral triangular path, with the three frames of coordinates S_0, S_1 and S_2, one at each vertex (coincident with S'_0, S'_1 and S'_2, not represented, for $t_{k\,0} = t'_{k\,0} = 0$), and the velocity vectors at the beginning of each leg of the path. For simplicity, the circumference is also not represented.

Take a regular polygon of n sides inscribed in a circle, immobile in the the plane xy (on the positive side of the y-axis) of the inertial frame S of observer A. Besides the variable n, there are two other constant basic data to take into account: $\beta = v/c$ and r (the *radius* of the circle). These parameters give:

- An angle at the center, $\theta = \frac{360°}{n}$;
- A length for each side, $l = 2r\sin\frac{\theta}{2}$;
- A constant time, $t = l/v$, to cover the distance between two consecutive vertices.

Now, another observer, B, in the origin of its proper frame S', is moving with velocity v (constant in modulus) along the sides of the polygon, abruptly changing trajectory at each vertex. Vertices are numbered consecutively from $\mathbf{0}$ (the starting point, where A stands still) to $\mathbf{n-1}$, an additional vertex $\mathbf{n}$ being coincident at the end with $\mathbf{0}$, closing the trajectory. For the first side of the path, from $\mathbf{0}$ to $\mathbf{1}$, the velocity $\mathbf{v}$ is aligned with the x-axis. When B reaches $\mathbf{1}$, he begins moving in a direction that makes an angle $\varphi_1 = \theta$ with the x-axis. This is a cumulative process, in such a way that, for the trajectory beginning at the vertex **k** (the **leg k+1**), we get

$$\varphi_k = k\,\theta \quad \Longrightarrow \quad \begin{cases} \beta_x = \beta\cos\varphi_k \\ \beta_y = \beta\sin\varphi_k . \end{cases} \qquad (6)$$

According to Lorentz transformations, excluding z coordinate,

$$\begin{pmatrix} ct' \\ x' \\ y' \end{pmatrix} = \boldsymbol{\Lambda}' \begin{pmatrix} ct \\ x \\ y \end{pmatrix}, \qquad (7)$$

Where $\boldsymbol{\Lambda}'$ is the inverse three-dimensional Lorentz matrix for a boost in an arbitrary direction $(\beta_x\,;\,\beta_y\,;\,\beta_z)$ [6]. Consequently, following (6), the 3×3 Lorentz k matrix [for the k leg of the path] assumes the form

$$\boldsymbol{\Lambda}'_k = \begin{pmatrix} \gamma & -\gamma\beta\cos\varphi_k & -\gamma\beta\sin\varphi_k \\ -\gamma\beta\cos\varphi_k & 1+(\gamma-1)\cos^2\varphi_k & (\gamma-1)\sin\varphi_k\cos\varphi_k \\ -\gamma\beta\sin\varphi_k & (\gamma-1)\sin\varphi_k\cos\varphi_k & 1+(\gamma-1)\sin^2\varphi_k \end{pmatrix}. \qquad (8)$$

In fact, we just need to focus on the time coordinate, that is, in the first-row matrix:

$$\boldsymbol{\Lambda}'^{(t)}_k = (\gamma \quad -\gamma\beta\cos\varphi_k \quad -\gamma\beta\sin\varphi_k\,). \qquad (9)$$

We deal here with a succession of coordinate frames, S_k and S'_k, all of them parallel to each other, consecutively reset and centered in each vertex k of the polygon. The variable time, in each leg of the path, will be restricted to the extremes: $t_0 = 0$ and $t = l/v$, establishing the convention that, for the **leg k** of the path (that is, between vertices $\mathbf{k-1}$ and **k**):

- x_i^k or $\mathrm{x'}_i^k$, written in normal lowercase, represent the matrix of coordinates of (vertex) i relatively to (vertex) k in "internal time" $t = l/v$;
- $\mathbf{x}_i^k$ or $\mathbf{x'}_i^k$, written in bold lowercase, represent the matrix of coordinates of i relatively to k in "internal time" $t_0 = 0$.

Note that this "internal time" refers to each frame S_k and S'_k; and that the cummulative final time in frame S_{k-1} [or S'_{k-1}] corresponds to the cummulative initial time in S_k [or S'_k]. Examining the problem as seen by the moving frame S', which has n successive 'components' S'_k (and this is just an option, the conclusion is the same if we consider the observer A moving in 'immobile' S'), we must determine, for each vertex **k**, the matrix of the so-called **summative coordinates** of vertex **0** in relation to **k**, which is defined as follows:

$$\mathbf{X'}_0^k = \mathbf{x'}_0^k + \mathbf{X'}_0^{k-1} \quad \text{along with } \mathbf{X'}_0^0 = 0\,, \quad \text{that is, } \mathbf{X'}_0^1 = \mathbf{x'}_0^1. \qquad (10)$$

This recursive definition leads to the ultimate objective, the final summative coordinates given by $\mathbf{X'}_0^n$. The precedent definition implies that:

$$\mathbf{X'}_0^n = (\mathbf{x'}_0^n + \mathbf{x'}_0^{n-1} + \ldots + \mathbf{x'}_0^1) = \textstyle\sum_{k=1}^{n} \mathbf{x'}_0^k, \quad (11)$$

Keeping in mind that

$$\mathbf{x'}_0^k = \boldsymbol{\Lambda}'_{k-1}\ \mathrm{x}_0^{k-1}. \qquad (12)$$

Keeping also in mind that the time required to cover each leg is always the same ($t = l/v$), one deduces that

$$\mathrm{x}_0^k = \begin{pmatrix} ct \\ r\big(\cos\alpha_0 - \cos(\alpha_0 + k\,\theta)\big) \\ r\big(\sin\alpha_0 - \sin(\alpha_0 + k\,\theta)\big) \end{pmatrix}, \quad \text{where} \quad \alpha_0 = -\left(90° + \frac{\theta}{2}\right). \qquad (13)$$

2.2. The Round Trip

The simplest example of this problem is the **linear round trip**, seen as a "two-sided" regular polygon. In this case, $\theta = \alpha_0 = 180°$ and $l = 2r$. The summative result, for the time coordinate is

$$\mathbf{T'}_0^2 = t\left(\gamma + \frac{1}{\gamma}\right)$$

We would get the same result if we considered the observer A moving in the frame of B; that means that $\mathbf{T}_0^2 = \mathbf{T'}_0^2$. In fact, the coexistence of both dilation and contraction factors assures the reciprocity of twins in the argument and the equivalence of inertial frames. So, both twins age the same.

2.3. The Challenges

Concerning the transformation of **time**, the equations (11) and (12) give:

$$\mathbf{T'}_0^n = \textstyle\sum_{k=1}^{n} \mathbf{t'}_0^k, \quad (14)$$

With

$$c\mathbf{t}'^{k}_{0} = \mathbf{\Lambda}'^{(t)}_{k-1}\, \mathrm{x}^{k-1}_{0}. \qquad (15)$$

Remark that, since $\varphi_0 = 0°$, the first term of the sequence $\mathbf{t}'^{k}_{0}$ is always given by

$$c\mathbf{t}'^{1}_{0} = \mathbf{\Lambda}'^{(t)}_{0}\, \mathrm{x}^{0}_{0} = (\gamma \quad -\gamma\beta \quad 0)\begin{pmatrix} ct \\ 0 \\ 0 \end{pmatrix} \;\Rightarrow\; \mathbf{t}'^{1}_{0} = \gamma t, \qquad (16)$$

Regardless of the value of n. For the last term in the sum for $\mathbf{T}'^{n}_{0}$, I verified in the calculated results presented in Luís [5] that it corresponds to $\mathbf{t}'^{n}_{0} = \frac{t}{\gamma}$. However, I have not been able then to prove it for $k > 2$; this will be the first goal to achieve here.

That said (and noting that for simplicity's sake, we are using the same symbol of the matrix $\mathbf{t}'^{k}_{0}$ for its unique content, as well as $\mathbf{T}'^{n}_{0} = \mathbf{T}'$), the sequence of the $\mathbf{t}'^{k}_{0}$ appears as a quite capricious one, though the equation (2),

$$\mathbf{T}' = \mathbf{T} = \frac{n}{2}\; t\left(\gamma + \frac{1}{\gamma}\right),$$

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$$\mathbf{T}' = \mathbf{T} = \frac{n}{2}\; t\left(\gamma + \frac{1}{\gamma}\right),$$

Expresses the sum of its terms as if they were the first n terms of a *decreasing arithmetic sequence*, with γt and t/γ as the first and the last terms respectively, and a common difference given by

$$q = -\frac{\gamma\, \beta^2 t}{n-1}.$$

For instance, concerning $\mathbf{t}'^{k}_{0}$ for $n = 10$ (making $\beta = 0.20$ and $r = 1000\ m$, so $t = 1.0308 \times 10^{-5}\ s$), we get, for $k = 1$ to $k = 10$, the following bizarre sequence:

$1.0520\,;\ 1.0861\,;\ 1.0991\,;\ 1.0861\,;\ 1.0520\,;\ 1.0099\,;\ \ 0.9759\,;\ 0.9629\,;\ 0.9759\,;\ 1.0099\ \ \times\ 10^{-5}$s.

Which sum is, indeed, $\mathbf{T}'^{10}_{0} = 5\, t\left(\gamma + \frac{1}{\gamma}\right) = 1.0310 \times 10^{-4}\ s$.

An immediate question arises: *what is the enigmatic relationship between the two sequences?* The answer comes from looking at the graph representing both, displayed in Figure 2. A symmetry is highlighted there, which can be mathematically expressed by the equation

$$\mathbf{t}'^{1+k}_{0} - \mathbf{t}'^{1}_{0} = \mathbf{t}'^{n}_{0} - \mathbf{t}'^{n-k}_{0}, \ (17)$$

or, equivalently,

$$\mathbf{t}'^{1+k}_{0} + \mathbf{t}'^{n-k}_{0} = \mathbf{t}'^{1}_{0} + \mathbf{t}'^{n}_{0}. \qquad (18)$$

These equations apply as well to the correspondent arithmetic progression and this is why both sequences conduce to the same final sum. Therefore, in order to demonstrate the validity of the conjecture formulated by (2),

$$\mathbf{T}' = \mathbf{T} = \frac{n}{2}\; t\left(\gamma + \frac{1}{\gamma}\right),$$

We just need to prove the validity of equation (18), together with $\mathbf{t}'^{n}_{0} = \frac{t}{\gamma}$. That is exactly what we will do in the sequence.

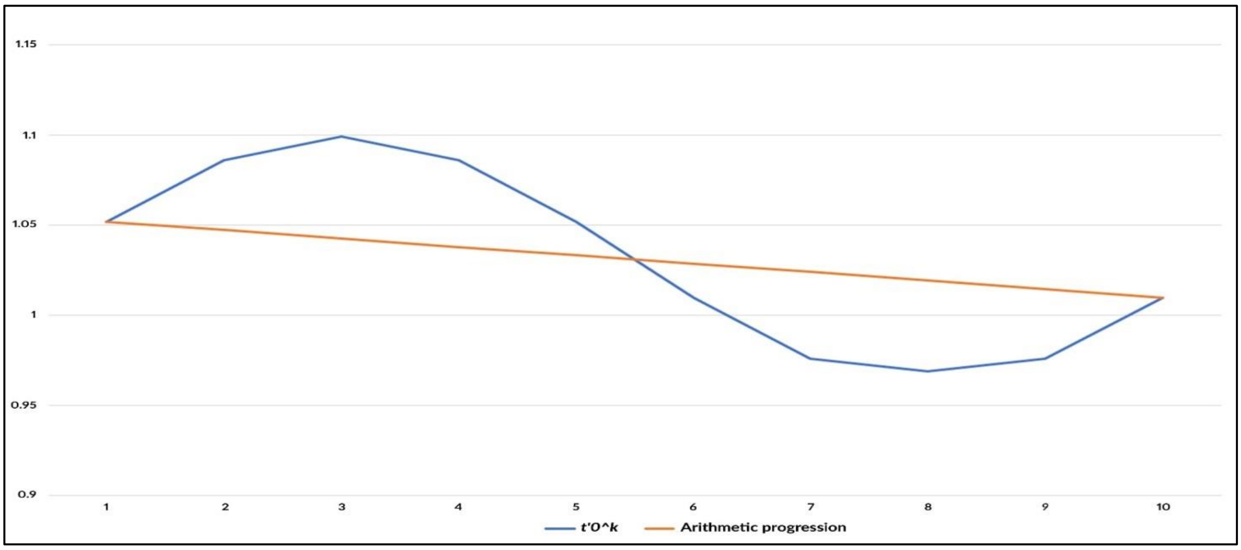

Figure-2. Comparative evolution, for $n = 10$, concerning the time sequence $\mathbf{t}'^{k}_{0}$ and the correspondent arithmetic progression

2.4. The Last Term

We wish to determine

$$\mathbf{t}'^{n}_{0} = \mathbf{\Lambda}'^{(t)}_{n-1}\ \mathrm{x}_0^{n-1}. (19)$$

Since

$$\varphi_{n-1} = 360° - \theta \quad \Rightarrow \quad \begin{cases} \cos\varphi_{n-1} = \cos\theta \\ \sin\varphi_{n-1} = -\sin\theta, \end{cases}$$

We get

$$\mathbf{\Lambda}'^{(t)}_{n-1} = (\gamma \quad -\gamma\beta\cos\theta \quad \gamma\beta\sin\theta)\,, (20)$$

As well as

$$\begin{cases} \cos(\alpha_0 + \varphi_{n-1}) = \cos(\alpha_0 - \theta) \\ \sin(\alpha_0 + \varphi_{n-1}) = \sin(\alpha_0 - \theta). \end{cases} \quad (21)$$

Hence,

$$\mathrm{x}_0^{n-1} = \begin{pmatrix} ct \\ r(\cos\alpha_0 - \cos(\alpha_0 - \theta)) \\ r(\sin\alpha_0 - \sin(\alpha_0 + k\,\theta)) \end{pmatrix}; \qquad (22)$$

and we get from (15):

$$\mathbf{t}'^{n}_{0} = \gamma ct - \gamma\beta r\left[\cos\theta(\cos\alpha_0 - \cos(\alpha_0 - \theta)) - \sin\theta(\sin\alpha_0 - \sin(\alpha_0 - \theta))\right]$$

which, after simplification, gives

$$\mathbf{t}'^{n}_{0} = \gamma\left[t - \frac{\beta r}{c}(\cos\alpha_0(\cos\theta - 1) - \sin\alpha_0\sin\theta)\right]. \qquad (23)$$

Now,

$$\frac{\beta r}{c}\cos\alpha_0 = -\frac{\beta}{c}\left(r\sin\frac{\theta}{2}\right) = -\frac{1}{2}\beta^2 t,$$

because $r\sin\frac{\theta}{2} = \frac{l}{2} = \frac{vt}{2}$. So, the equation (23) turns into

$$\begin{aligned} \mathbf{t}'^{n}_{0} &= \gamma t\left[1 + \frac{1}{2}\beta^2(\cos\theta - 1) + \frac{\beta r}{ct}\sin\alpha_0\sin\theta\right] \\ &= \gamma t\left[\left(1 - \frac{1}{2}\beta^2\right) + \frac{1}{2}\beta^2\left(\cos\theta + \frac{2r}{vt}\sin\alpha_0\sin\theta\right)\right] \end{aligned}$$

The last expression between brackets may be resolved as follows:

$$R = \cos\theta + \frac{2r}{vt}\sin\alpha_0\sin\theta = \cos\theta + \frac{\sin\alpha_0\sin\theta}{\sin\frac{\theta}{2}};$$

But

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

and so, remembering that $\sin\alpha_0 = -\cos\frac{\theta}{2}$,

$$R = \cos\theta + 2\sin\alpha_0\cos\frac{\theta}{2} = \cos\theta - 2\cos^2\frac{\theta}{2}.$$

On the other hand,

$$\begin{aligned} \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 \\ \Rightarrow\ 2\cos^2\frac{\theta}{2} = \cos\theta + 1 \\ \Rightarrow\ R = \cos\theta - (\cos\theta + 1) = -1. \end{aligned}$$

Finally, then,

$$\mathbf{t}'^{n}_{0} = \gamma t(1 - \beta^2),$$

That is,

$$\mathbf{t}'^{n}_{0} = \frac{t}{\gamma}. (24)$$

2.5. The Equation (27)

Let us prove now the validity of equation (18),

$$\mathbf{t}'^{1+k}_{0} + \mathbf{t}'^{n-k}_{0} = \mathbf{t}'^{1}_{0} + \mathbf{t}'^{n}_{0}.$$

First, concerning $\mathbf{t}'^{1+k}_{0}$, following (15):

$$c\mathbf{t}'^{1+k}_{0} = \mathbf{\Lambda}'^{(t)}_{k}\ \mathrm{x}_0^{k},$$

Where, as we have seen [eq. (9)],

$$\mathbf{\Lambda}'^{(t)}_{k} = (\gamma \quad -\gamma\beta\cos\varphi_k \quad -\gamma\beta\sin\varphi_k)$$

and [eq. (13)]

$$\mathrm{x}_0^{k} = \begin{pmatrix} ct \\ r(\cos\alpha_0 - \cos(\alpha_0 + \varphi_k)) \\ r(\sin\alpha_0 - \sin(\alpha_0 + \varphi_k)) \end{pmatrix}.$$

This means that

$$c\mathbf{t}'^{1+k}_0 = \gamma ct - \gamma\beta r\left[\cos\varphi_k\left(\cos\alpha_0 - \cos(\alpha_0+\varphi_k)\right) + \sin\varphi_k\left(\sin\alpha_0 - \sin(\alpha_0+\varphi_k)\right)\right]$$

or, simplifying the expression in right brackets and dividing both members by c,

$$\mathbf{t}'^{1+k}_0 = \gamma t - \frac{\gamma}{c}\beta r[\cos\alpha_0(\cos\varphi_k - 1) + \sin\alpha_0\sin\varphi_k].$$

$$\begin{aligned}\mathbf{t}'^{1+k}_0 &= \gamma t - \frac{\gamma}{c}\beta r[\cos\alpha_0(\cos\varphi_k - 1) + \sin\alpha_0\sin\varphi_k] \\ &= \gamma t - \frac{\gamma}{c}\beta r\cos\alpha_0[(\cos\varphi_k - 1) + \tan\alpha_0\sin\varphi_k]\end{aligned}$$

But

$$-\frac{\gamma}{c}\beta r\cos\alpha_0 = \frac{\gamma}{c}\beta\left(r\sin\frac{\theta}{2}\right) = \frac{\gamma\beta}{c}\frac{vt}{2} = \frac{\gamma\beta^2 t}{2}.$$

So,

$$\mathbf{t}'^{1+k}_0 = \gamma t\left[\left(1 - \tfrac{1}{2}\beta^2\right) + \tfrac{1}{2}\beta^2(\cos\varphi_k + \tan\alpha_0\sin\varphi_k)\right]. \qquad (25)$$

Note that this equation is valid for whatever k (with $0 \le k < n$), so it can be applied to $\mathbf{t}'^{n-k}_0$, with the understanding that we must put $n-(k+1)$ in the place of k. Besides, considering that

$$\varphi_{n-(k+1)} = 360° - \varphi_{k+1} \quad \Rightarrow \quad \begin{cases}\cos\varphi_{n-(k+1)} = \cos\varphi_{k+1} \\ \sin\varphi_{n-(k+1)} = -\sin\varphi_{k+1},\end{cases}$$

one gets

$$\mathbf{t}'^{n-k}_0 = \gamma t\left[\left(1 - \tfrac{1}{2}\beta^2\right) + \tfrac{1}{2}\beta^2(\cos\varphi_{k+1} - \tan\alpha_0\sin\varphi_{k+1})\right]. \qquad (26)$$

Now, adding the two partial times, given by (25) and (26), it yields

$$\mathbf{t}'^{1+k}_0 + \mathbf{t}'^{n-k}_0 = 2\gamma t\left(1 - \frac{1}{2}\beta^2\right) + \frac{\gamma t}{2}\beta^2[(\cos\varphi_k + \cos\varphi_{k+1}) + \tan\alpha_0(\sin\varphi_k - \sin\varphi_{k+1})].$$

Remark that

$$2\gamma t\left(1 - \frac{1}{2}\beta^2\right) = 2\gamma t - \gamma t\beta^2 = \gamma t + \frac{t}{\gamma} = \mathbf{t}'^{1}_0 + \mathbf{t}'^{n}_0 \ ;$$

as a consequence,

$$\begin{aligned}\mathbf{t}'^{1+k}_0 + \mathbf{t}'^{n-k}_0 &= \mathbf{t}'^{1}_0 + \mathbf{t}'^{n}_0 + \\ &+ \frac{\gamma t}{2}\beta^2[(\cos\varphi_k + \cos\varphi_{k+1}) + \tan\alpha_0(\sin\varphi_k - \sin\varphi_{k+1})].\end{aligned}$$

So, for it to be

$$\mathbf{t}'^{1+k}_0 + \mathbf{t}'^{n-k}_0 = \mathbf{t}'^{1}_0 + \mathbf{t}'^{n}_0$$

it is enough to prove that the expression between right brackets is null – which is true, as we will see. Let us put

$$\mathrm{A} = \cos\varphi_k + \cos\varphi_{k+1}$$

and

$$\mathrm{B} = \tan\alpha_0(\sin\varphi_k - \sin\varphi_{k+1}),$$

so that

$$\mathbf{t}'^{1+k}_0 + \mathbf{t}'^{n-k}_0 = \mathbf{t}'^{1}_0 + \mathbf{t}'^{n}_0 + \frac{\gamma t}{2}\beta^2(\mathrm{A} + \mathrm{B}).$$

On the one hand,

$$\begin{aligned}\mathrm{A} &= \cos\varphi_k + \cos(\varphi_k + \theta) = \\ &= \cos\varphi_k + (\cos\varphi_k\cos\theta - \sin\varphi_k\sin\theta) = \\ &= \cos\varphi_k(1 + \cos\theta) - \sin\varphi_k\sin\theta \\ &= \mathrm{A1} + \mathrm{A2}.\end{aligned}$$

On the other hand, concerning the component B,

$$\begin{aligned}\sin\varphi_k - \sin\varphi_{k+1} &= \sin\varphi_k - \sin(\varphi_k + \theta) = \\ &= \sin\varphi_k - (\sin\varphi_k\cos\theta + \cos\varphi_k\sin\theta) = \\ &= \sin\varphi_k(1 - \cos\theta) - \cos\varphi_k\sin\theta.\end{aligned}$$

Now, $\tan\alpha_0 = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$, and so, because

$$\cos\varphi_k\sin\theta = 2\cos\varphi_k\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

one gets:

$$\begin{aligned}\mathrm{B} &= \mathrm{B1} + \mathrm{B2} \\ &= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\sin\varphi_k(1 - \cos\theta) - 2\cos^2\frac{\theta}{2}\cos\varphi_k\,.\end{aligned}$$

But $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, and therefore

$$\mathrm{B2} = -2\cos^2\frac{\theta}{2}\cos\varphi_k = -\cos\varphi_k(1 + \cos\theta) = -\mathrm{A1}.$$

In conclusion,

$$\mathrm{A}+\mathrm{B}=\mathrm{A2}+\mathrm{B1}=\sin\varphi_k\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}(1-\cos\theta)-\sin\theta\right);$$

and so, for $\mathrm{A}+\mathrm{B}=0$, it just needs to be

$$\cos\frac{\theta}{2}(1-\cos\theta)=\sin\frac{\theta}{2}\sin\theta.$$

Indeed, if we raise both terms to the square, we get

$$\cos^2\frac{\theta}{2}(\cos^2\theta+1-2\cos\theta)=\sin^2\frac{\theta}{2}\sin^2\theta$$

$$\cos^2\frac{\theta}{2}(-\sin^2\theta+2-2\cos\theta)=\sin^2\frac{\theta}{2}\sin^2\theta$$

$$2\cos^2\frac{\theta}{2}(1-\cos\theta)=(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2})\sin^2\theta=\sin^2\theta.$$

But, making, once again, $2\cos^2\frac{\theta}{2}=1+\cos\theta$, we finally get

$$1-\cos^2\theta=\sin^2\theta \quad\Rightarrow\quad \sin^2\theta+\cos^2\theta=1,$$

which is a fundamental trigonometric identity; so, $\mathrm{A}+\mathrm{B}=0$ and this finally proves that

$$\mathbf{t'}_0^{1+k}+\mathbf{t'}_0^{n-k}=\mathbf{t'}_0^{1}+\mathbf{t'}_0^{n},$$

and, consequently, the validity of equation (2).

3. Circular Path

This is basically a consolidated review of material from the previous article, Luís [5] . As noted there, the solution for polygons may be extended to circles by noting that equation (2) may be written as

$$\mathbf{T}=\mathbf{T}'=n\,t\,\delta_0=\delta_0\frac{nl}{v},\quad \text{or}\quad \mathbf{T}=\mathbf{T}'=\delta_0\frac{\mathrm{p}}{v},\qquad (27)$$

Where $\mathrm{p}=nl$ is the Euclidean perimeter of the polygonal line. The correspondent *relativistic perimeter*, $\mathrm{p}_R=\mathrm{p'}_R$, is given by

$$\mathrm{p}_R=v\mathbf{T}\quad\Rightarrow\quad \mathrm{p}_R=\mathrm{p'}_R=\delta_0\mathrm{p}.\qquad (28)$$

One easily concludes that the polygon 'constructed' by the movement of B – over a regular one in the 'immobile' *Euclidean substratum* of A's reference frame – results **irregular** in B's proper frame. But, in the end, it comes that

$$\sum_{k=1}^{n}\mathbf{l'}^k=v\sum_{k=1}^{n}\mathbf{t'}_0^k=v\,\mathbf{T'}_0^n=\mathrm{p}_R,$$

That is, the relativistic perimeter of the polygon.

It can be shown that the irregularity progressively smooths out as the number of sides increases, until it disappears at the infinite limit, that is, in the case of the circumference. Therefore, a circle appears as a circle in both frames.

Now, once proved the validity of the conjecture formalized by (2), we are allowed to confidently extend (27) and (28) to the **circumference**. As a matter of fact, these equations establish $\mathbf{T}$ and p_R as independent from the number of sides of the polygon. So, they must remain valid in the limit $n\to\infty$, that is, when the polygon turns into a circumference, yielding

$$\begin{cases}\mathbf{T}=\delta_0\frac{2\pi r}{v}\\ \mathrm{p}_R=\delta_0\,2\pi r.\end{cases}\qquad (29)$$

Furthermore, as was shown in the previous article, the *relativistic radius* of the circumference is given by

$$\mathrm{r}_R=\delta_0\,r,\qquad (30)$$

Where r_R is the *relativistic radius* of the circumference, for a correspondent Euclidian radius r. So,

$$\mathrm{p}_R=2\pi\,\mathrm{r}_R,\qquad (31)$$

An equality that is independent from the velocity of B, thus preserving the Euclidean proportion.

4. Conclusion

This paper brings the last stone to resolve an issue dating from the foundation of special relativity: there is no Twin's Paradox at all, whether on a round trip, or on a closed polygonal or circular path.

Taking up a critical reappraisal of this subject, now in relation to polygonal or circular paths, the validity of

$$\mathbf{T}'=\mathbf{T}=\frac{n}{2}\;t\left(\gamma+\frac{1}{\gamma}\right),$$

has been proved, for the total time required for both twins to meet again, has been proved (along with $\mathbf{t'}_0^n=\frac{t}{\gamma}$). Therefore, together with physical justifications, these results are no longer conjectural, as presented in [5], but enunciate scientific facts. The same criterion applies to two interrelated dilation factors, also established in the precedent paper:

$$\delta_0=\frac{1}{2}\left(\gamma+\frac{1}{\gamma}\right)\quad\text{and}\quad \delta=\gamma\delta_0=\frac{1}{2}(\gamma^2+1).$$

In principle, these conclusions should be experimentally verifiable, for example in particle accelerators, thus confirming or invalidating them.

5. Statements and Declarations

5.1. Conflict of Interest Statement

The author has no conflicts to disclose.

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